Calhoun
Institutional Archive of the Naval Postgraduate School

Calhoun: The NPS Institutional Archive

DSpace Repository

Theses and Dissertations 1. Thesis and Dissertation Collection, all items

1961

Correlation processes in antenna arrays.

Linder, Isham Wiseman

University of California, Berkeley

http://hdl.handle.net/10945/11878

Downloaded from NPS Archive: Calhoun

DUDLEY KNOX LIBRARY

Calhoun is the Naval Postgraduate School's public access digital repository for research materials and institutional publications created by the NPS community. Calhoun is named for Professor of Mathematics Guy K. Calhoun, NPS's first appointed -- and published -- scholarly author.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

http://www.nps.edu/library

NPS ARCHIVE
1961
LINDER, I.

CORRELATION PROCESSES IN ANTENNA ARRAYS

ISHAM W. LINDER

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY CA 93943-5101

1961

Correlation Processes in Antenna Arrays

By

Isham Wiseman Linder

B.S. (United States Naval Academy) 1946
B.S. (Massachusetts Institute of Technology) 1949
M.S. (United States Naval Postgraduate School) 1956
DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Science

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA

Library
U. S. Naval Postgraduate School
Monterey, California

CORRELATION PROCESSES IN ANTENNA ARRAYS

TABLE OF CONTENTS

I. Introduction 1

II. Antenna Patterns of Nonlinear Arrays 4
  A. General 4
  B. Monochromatic Signals 6
  C. Band-Limited Random Signals, General Case 10
  D. Band-Limited Random Signals, Specific Cases 19

III. Effect of Finite Averaging Time 25

IV. Resolution Characteristics of Correlation Arrays 35

V. Effect of Distributed Noise on the Nonlinear Antenna System 43

VI. Antenna Arrays and Channel Capacity 52

VII. Design of Nonlinear Arrays 56
  A. Basic Considerations 56
  B. Products of Linear Arrays 57
  C. Data Processing by Correlation Circuits 59

Appendix I. The Probability Distribution for the Output Noise Voltage of a Two-Element Correlation Array 67

Appendix II. Calculation of Voltage Response Patterns with Two Signal Sources 74

Appendix III. Application of Correlation Techniques to Linear Arrays 81

References 108

CORRELATION PROCESSES IN ANTENNA ARRAYS

I. INTRODUCTION

The design of an antenna array whose performance must be optimized in some specific sense (maximum signal to noise ratio in the output circuit, minimum antenna pattern beamwidth, etc.) when the array is to receive a signal buried in a general noisy medium can be accomplished through noise theory considerations and correlation techniques. These statistical methods can be applied to two general types of multi-element antenna arrays. First, there is the normal linear array in which the output circuit contains the sum of the voltages induced on the individual elements. Analysis of the noise voltages in terms of their correlation coefficients will lead to a specification of element spacing which will optimize the antenna performance. Second, there is the newer nonlinear array in which both multiplication and summation processes may be used in combining the individual element voltages to form a single array output voltage. This type of array results from an attempt to implement the definition of the correlation coefficient by physical circuitry, and can be extensively analyzed by methods of statistical noise theory.

Correlation techniques have been introduced into the analysis of the linear array primarily by acoustical engineers[1,2].

[1,2] References are listed at the end of this report.

The manner in which the correlation coefficient enters into a consideration of this general type of array can be seen from the simple example of the two-element linear array. A random noise source will induce a voltage on one element which will be identical to that induced on the other element except for a time delay caused by the time of travel of the wavefront between elements. If $e_1(t)$ and $e_2(t)$ represent the instantaneous voltages on the two elements, then the average power in the output circuit of the array becomes:

$$\overline{e^2 = [e_1(t_1) + e_2(t_2)]^2}$$

$$= \overline{e_1(t_1)^2} + \overline{e_2(t_2)^2} + 2\,\overline{e_1(t_1)e_2(t_2)}$$

$$= \overline{e_1(t_1)^2} + \overline{e_2(t_2)^2} + 2\,R_{12}(t_1 - t_2)$$

where $R_{12}(\tau)$ is the cross-correlation coefficient.

The cross-correlation coefficient will, in general, take on both positive and negative values as τ is varied. Thus it should be possible to select an element spacing which will minimize the average noise power, $\overrightarrow{e^2}$. This rather simple idea can be applied to multi-element linear arrays -- the output voltage is expressed as the sum of noise power contributions of each element plus the cross correlation coefficients of all pairs of elements in the array. This total average noise power is then minimized by proper spacing of the antenna elements.

Some previous results in this application of correlation

techniques to linear arrays are presented in Appendix I, together with a general methodology for applying these principles to the types of antennas useful in electromagnetic radiation.

The main body of this report will be concerned with the analysis of the second type of array--the array which utilizes multiplicative processes in combining the individual element voltages to form the single array output voltage. The initial section will be concerned with a review of antenna response patterns which result from these nonlinear processes in the antenna circuitry. Succeeding sections will discuss the effects of distributed noise, of multiple signal sources, and of a variable averaging time on the output voltage. Finally the results of these analyses will be utilized to reach general conclusions about the type of nonlinear antenna which will give optimum performance, and examples of these antennas will be developed.

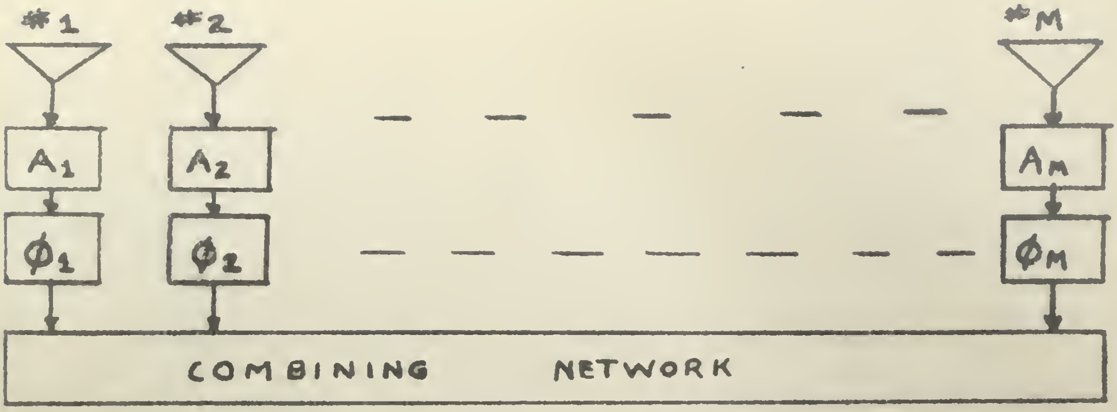
II. ANTENNA PATTERNS OF NONLINEAR ARRAYS

A. General

The principle of multiplying together the output voltages from two antennas is well known; its applications include, for example, radio astronomy. Several papers [3,4,5] have discussed some aspects of the subject, but no treatment appears to have been made of the problem of development of nonlinear arrays intended to detect a signal buried in a medium of randomly varying noise sources or a signal partially hidden by other undesired signals.

Before discussing the types of antenna directional patterns resulting from the use of nonlinear processes in the antenna circuitry, it would be best to define the general type of antenna array which will be employed and to establish the general description of the randomly varying noise and signal sources.

To remain completely general, it will be assumed that the noise and signal voltages will be applied to the M elements of a receiving array, and that these elements are connected to individual amplifiers (A_1, A_2, - - -, A_M) and phase shifters (ϕ_1, ϕ_2, - - -, ϕ_M). The output of each antenna-amplifier-phase shifter circuit is connected to a combining network which will operate on the individual element voltages to provide an optimum output voltage from the total array.



The combining network will consist of various summation and multiplication circuits. For practical effectiveness time averaging will be required in this network. The combination of multiplication and time averaging then satisfies the definition of the correlation function:

$$R_{ij}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} n_i(t)\, n_j(t+\tau)\, dt.$$

In considering practical networks that can be used to carry out this correlation process, two general variables arise: (1) possible combinations of the multiplication and averaging circuits, and (2) restriction of time averaging to a finite time interval. Both of these will be considered in subsequent sections, and will form the basis for the final statement of the optimum type of nonlinear array.

For this analysis assume that the noise sources are spherically distributed and are statistically independent. The voltage produced on an isotropic antenna by the noise originating in an element of solid angle $d\Omega$ can be represented by the usual Fourier series:[6,7]

$$x(\tau) = \sum_{k=0}^{\infty} [a_k \cos(2\pi f_k t) + b_k \sin(2\pi f_k t)\ \ ]$$

where

$$\lim_{T\to\infty} T\cdot E\left\{a_m a_n\right\} = 0, \qquad \text{for } m \neq n$$

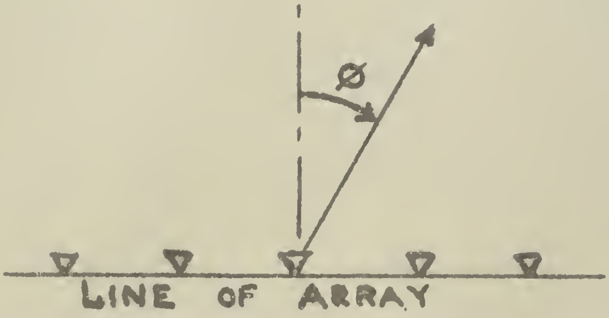
$$\lim_{T\to\infty} T\cdot E\left\{b_m b_n\right\} = 0, \qquad \text{for } m \neq n$$

$$\lim_{T\to\infty} T\cdot E\left\{a_m b_n\right\} = 0$$

$$\lim_{T\to\infty} T\cdot E\left\{a_n{}^2\right\} = \lim_{T\to\infty} T\cdot E\left\{b_n{}^2\right\} = 2\,W(f_n)$$

In this representation, the coefficients a_k and b_k are distributed normally with means zero; E is the expectation operator, and W(f) is the (two-sided) power spectral density of the voltage induced on the antenna elements by the source. It should be noted that the inherent frequency response of the antenna elements and amplifier circuits will lead to the familiar narrow-band gaussian random process.

The directivity pattern of the antenna array will be defined to be the voltage response at the output of the antenna array for an arbitrary angular location, ϕ, of a single signal source; for simplicity this will be the two-dimensional pattern.



B. Monochromatic Signals.

The antenna directivity patterns which result when

individual element voltages induced by a single monochromatic source (negligible noise) are correlated have been investigated by several researchers.[3,4] In general, the procedure has been to recall the directivity patterns of linear additive arrays and then to demonstrate product arrays which will give equivalent directivity patterns.

The directivity patterns for linear arrays with constant element spacing, d, can be expressed by one of two series, depending on an odd or an even number of elements in the array.

1. For an array of 2n+1 elements, the sum pattern is given by

$$P_{2n+1}(\phi) = \sum_{k=0}^{n} A_k \cos 2ku$$

where $u = (\pi d/\lambda) \sin \phi$. The array is assumed to be symmetrical with respect to its center element, A_0, $(A_i = A_{-i})$. $P_{2n+1}(\phi)$ is the amplitude gain given by the array to an input signal $E_0 \sin(\omega t+\psi)$. The center of the array is the phase reference.

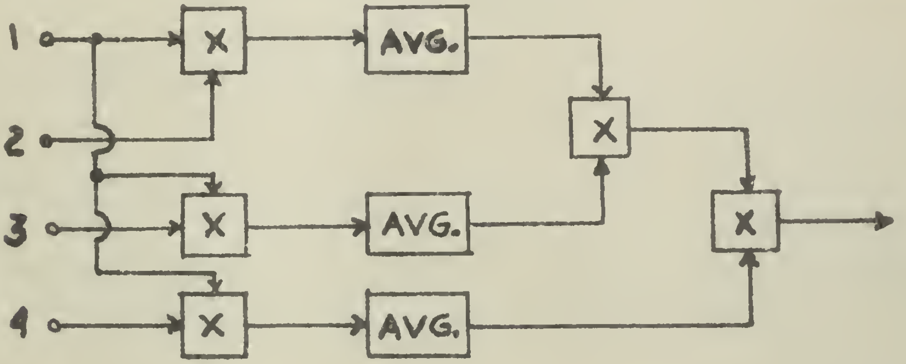
2. A linear array of 2n elements and constant spacing, d, has a sum pattern

$$P_{2n}(\phi) = \sum_{k=1}^{n} B_k \cos(2k - 1)u$$

Again symmetry is assumed, so B_k is the common amplification of the two elements which have distance $(k - \frac{1}{2})$ d from the

array center.

Berman and Clay have described an array in which pairs of signals are multiplied together and then time averaged. The output voltage is formed by multiplying together all of the resulting time averages. With this procedure, an array of n + 1 elements, with successive elements spaced D, 2D, 4D, . . . , $2^{n-1}D$ from the first element, has a directivity pattern equivalent to that of a 2^n element additive array with constant element spacing 2D. In this case the product array is about one-fourth the length of the equivalent linear array. Such a product array with four elements could be represented as

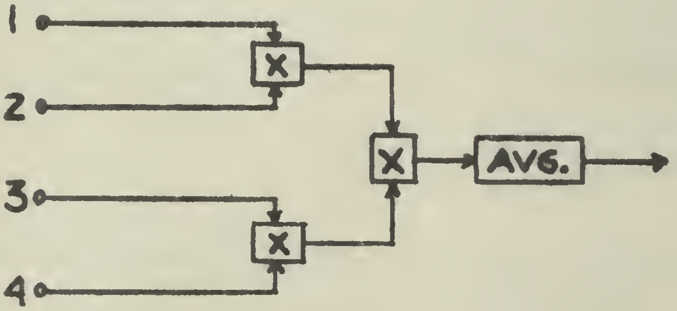


$$V_{out} \propto \prod_{i=2}^{4} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos\omega t \quad \cos(\omega t - \omega\tau_{1i})\, dt$$

$$V_{out} \propto \prod_{i=2}^{4} \cos \omega\tau_{1i}$$

With proper spacing of the elements, this pattern is the same as that of an eight-element linear array.

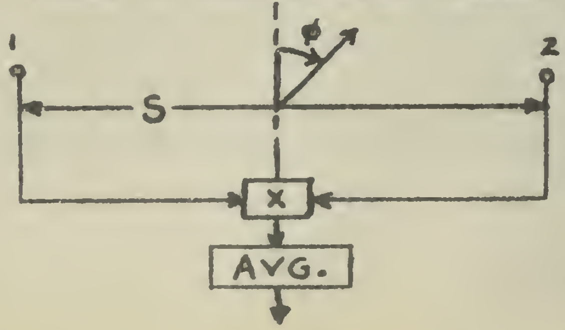
A second possible product array carries out all desired multiplications before finally time averaging the product. Analysis of the directivity pattern in this case shows a smaller saving in overall length over the equivalent linear array than was the case in the first product array discussed. Again a four-element example is



$$V_{out} \propto \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \prod_{i=1}^{4} \cos(\omega t - \omega \tau_{1i})\,dt.$$

With proper spacing of the elements, this pattern is the same as that of a six-element linear array.

It is also possible to start with two elements and to process the element voltages to get directional patterns which are equivalent to those of odd or even element additive arrays.



Voltage induced on element #1 $\sin(\omega t + \psi)$

Voltage induced on element #2 $\sin[\omega t - (\frac{\omega s}{c} \sin\phi) + \psi]$

Output of multiplier $Y = \sin(\omega t + \psi)\sin[\omega t - (\frac{\omega s}{c} \sin\phi) + \Psi]$

After averaging $\overline{Y} = \frac{1}{2} \cos (\frac{\omega s}{c} \sin\phi)$

And, letting $d = 2s$, $\overline{Y} = \frac{1}{2} \cos(\frac{\pi d}{\lambda} \sin\phi) = \frac{1}{2} \cos u$

Brown and Rowlands have shown how to use this function to synthesize a directivity pattern. Using the relation

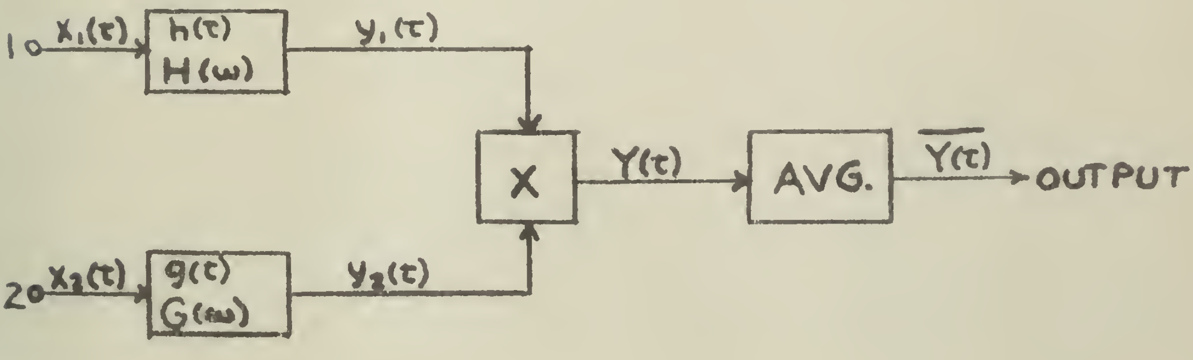
$$T_n(\cos u) = \cos nu \qquad n = 0, 1, 2, \ldots$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind, every term in the cosine series of the desired directivity pattern can be synthesized by a suitable combination of power law devices operating on the output of the two-element correlation pair. Arithmetically, two elements can thus be made to give a directivity pattern equivalent to that of a linear additive array of any arbitrary length. In practice, noise considerations, ignored in these pattern calculations, and the presence of more than one signal source would limit the length of the equivalent linear array which could be synthesized. These points will be considered in more detail in a later section.

C. Band-limited Random Signals, General Case

As with the monochromatic signals, the basic unit in the study of correlation of band-limited random signals is the

two-element array. In this case the signal can be described only in statistical terms, and statistical methods can be utilized to describe the performance of the system. As before, noise will be neglected in initial considerations and will enter in a later section. Correlation in such a two-element array is directly analogous to the operation of correlation detectors, which have been described in the literature.[8] Performance of correlation detectors, however, is usually studied in terms not directly applicable to the description of the characteristics of antenna arrays.



We assume that the voltages induced on the antenna elements are stationary and ergodic random functions of time with normal distributions and are confined to $2\pi B$, a closed interval in ω. The amplifiers are linear, time-invariant networks with system functions $H(\omega) = |H(\omega)| \; e^{j\eta(\omega)\tau}$ and $G(\omega) = |G(\omega)| e^{j\gamma(\omega)\tau}$, Fourier transformable into the impulse responses $h(t)$ and $g(t)$.

For each set of the ensemble of signal functions, the Fourier expansion is:

$$x_1(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \phi_i)$$

$$x_2(t) = \sum_{i=1}^{N} b_i \cos(\omega_i t + \psi_i)$$

where

1. a_i, b_i are Rayleigh distributed with $\overline{a_i^2} = 2W_1(f_1)\,\delta f$ and $\overline{b_i^2} = 2W_2(f_i)\,\delta f$, with $W(f_i)$ as the power density over a frequency interval δf centered at f_i;

2. ϕ_i, ψ_i have uniform distributions over $(0, 2\pi)$;

3. ω_0 is the lower edge of the band of width B cps, and $\omega_i = \omega_0 + 2\pi i/T;\ i = 1, \ldots, N.$

Then

$$y_1(t) = \sum_{i=1}^{N} a_i h_i \cos(\omega_i t + \phi_i + \eta_i)$$

$$y_2(t) = \sum_{i=1}^{N} b_i g_i \cos(\omega_i t + \psi_i + \gamma_i)$$

After multiplication

$$Y(t) = y_1(t)y_2(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i b_j g_j h_i \Big[\tfrac{1}{2}\cos(\omega_i t + \omega_j t + \phi_i + \psi_j + \eta_i + \gamma_j) + \tfrac{1}{2}\cos(\omega_i t - \omega_j t + \phi_i - \psi_j + \eta_i - \gamma_j)\Big].$$

The first cosine term describes a band of frequencies centered

at twice the mid-frequency of the band-limited signal, and the second cosine term describes the dc and low frequency component of the output voltage.

After averaging:

$$\overline{Y(t)} = \int_0^\infty \overline{a\ b\ g\ h} \cos(\phi - \psi + \eta - \gamma)\, d\omega .$$

Now in this antenna case $x_2(t) = x_1(t)\, e^{j\omega\tau}$. So $a_i = b_i$ and $\psi_i = \phi_i + \omega_i \tau$. And if the amplifiers are assumed to have identical phase functions and constant amplitude functions, the output is

$$\overline{Y(t)} = (G\ H) \int_0^\infty W(\omega) \cos \omega\tau\, d\omega = (G\ H)\, R(\tau).$$

of the correlation function. For this normally distributed signal with a rectangular frequency function (center frequency f_0)

$$\overline{Y(t)} = (G \cdot H) B \frac{\sin \pi B \tau}{\pi B \tau} \cos 2\pi f_0 \tau$$

The obvious advantage to specifying amplifiers with identical phase functions and constant amplitude functions is that this leads directly to a statement of the output of the system in terms of the correlation function of the input band-limited signal. A signal which is normally distributed over

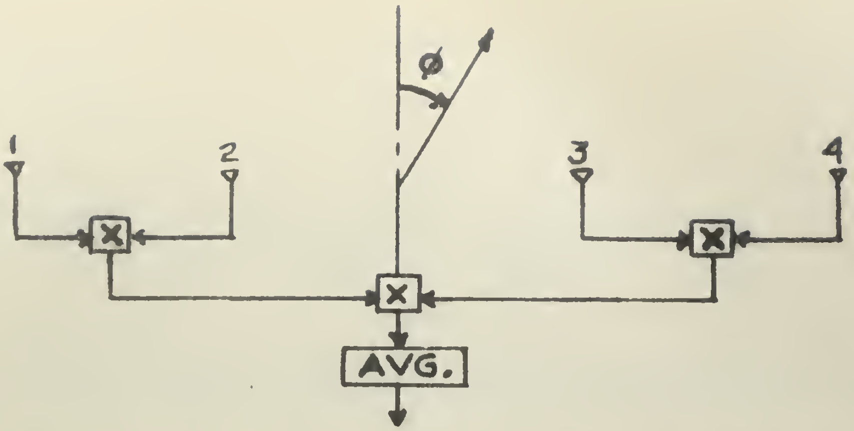
the pass band of the amplifiers can be analyzed most simply through its characteristic function. As a result of these two statements, the performance of correlation arrays with more elements can be determined in a direct manner.

Before considering more elements, however, it is useful to point out the self noise or fluctuation component of the output voltage of the multiplier in the simple two-element correlator. For a normally distributed signal source located in the principal lobe of the two-element array ($\tau = 0$), the output of the multiplier has a Chi-square distribution whose mean value is the desired array output voltage, and whose ac component is the self noise which must be minimized by time averaging. For an arbitrarily located source it was shown previously that the mean value is proportional to $R(\tau)$, the correlation coefficient. The variance is proportional to $[R(0)^2 + R(\tau)^2]$, or

$$\sigma^2 \propto R(\tau)^2 \left[1 + \frac{R(0)^2}{R(\tau)^2} \right]$$

This variance does not go to zero. Its value decreases by 1/2 as the signal moves from the main lobe to a position where its mean value is zero.

As an example of the calculations of directivity patterns of correlation arrays with a greater number of elements, consider a four-element array.



The desired output voltage is simply the expected value of the product of the four element voltages.

$$\overline{Y(t)} = \overline{V_1(t)V_2(t)V_3(t)V_4(t)}$$

And, since these are normally distributed band-limited voltages, this becomes

$$Y(t) = \left[V_1(t)V_2(t)\right]\left[V_3(t)V_4(t)\right] + \left[V_1(t)V_3(t)\right]\left[V_2(t)V_4(t)\right] +$$

$$\left[V_1(t)V_4(t)\right]\left[V_2(t)V_3(t)\right].$$

$$\propto R(\tau_{12})\, R(\tau_{34}) + R(\tau_{13})\, R(\tau_{24}) + R(\tau_{14})\, R(\tau_{23}).$$

With a rectangular frequency function, each correlation coefficient has the form

$$R(\tau_{ij}) = R(0)\,\frac{\sin \pi B \tau_{ij}}{\pi B \tau_{ij}} \cos 2\pi f_0 \tau_{ij}$$

And with relative element spacing chosen to make $d_{12} = D$, $d_{23} = D$, and $d_{34} = 2D$, the mean output voltage can be put in a

form resembling the directivity pattern of a uniform linear array.

$$Y(t) \propto \frac{1}{2}\left[\frac{\sin BX}{BX}\,\frac{\sin 2BX}{2BX} + \frac{\sin 2BX}{2BX}\,\frac{\sin 3BX}{3BX}\right]\cos 1(2f_0X) +$$

$$\frac{1}{2}\left[\frac{\sin BX}{BX}\,\frac{\sin 2BX}{2BX} + \frac{\sin BX}{BX}\,\frac{\sin 4BX}{4BX}\right]\cos 3(2f_0X) +$$

$$\frac{1}{2}\left[\frac{\sin BX}{BX}\,\frac{\sin 4BX}{4BX} + \frac{\sin 2BX}{2BX}\,\frac{\sin 3BX}{3BX}\right]\cos 5(2f_0X)$$

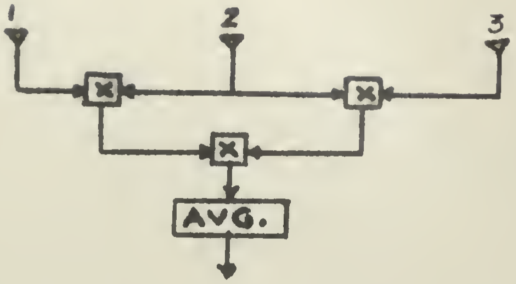
where $X = (\pi D/c)\sin\phi$. For a very narrow bandwidth the coefficients of the cosine terms approach unity and the directivity pattern of this correlation array (length 4D) is the same as that of a six-element uniform linear array (length 10D). As the bandwidth increases, these coefficients influence the amplitude of the sidelobe structure of the directivity pattern. This is shown in Fig. 1 for one basic spacing, D, and for two bandwidths.

The assumption of a random signal source with a normal distribution permits a pattern analysis by the characteristic function method. The characteristic function is

$$F_x(\xi) = E\left\{e^{j\xi_1 x_1 + j\xi_2 x_2 + \,\ldots\ldots\, + j\xi_K x_K}\right\} = \exp\left[-\frac{1}{2}\sum_{r-1}^{K}\sum_{s-1}^{K} R(\tau_{rs})\xi_r\xi_s\right].$$

and the pattern will be given by the appropriate coefficient in the expansion of this function. For example, consider a simple three-element array. The directivity pattern will be given by

$E\left\{V_1 V_2^2 V_3\right\}$,



where $E\left\{x\right\}$ is the expectation operator. This will be the coefficient of $[(j\xi_1)/1 \cdot (j\xi_2)^2/2. \cdot (j\xi_3)/1]$ in the power series expansion of $\exp[-(1/2)\sum_{r=1}^{3}\sum_{s=1}^{3} R(\tau_{rs})\xi_r \xi_s]$

$$E\left\{V_1 V_2^2 V_3\right\} \propto R(0)R(\tau_{13}) + 2R(\tau_{12})R(\tau_{23})$$

If we assume a rectangular frequency function and relative element spacings $d_{12} = D$, $d_{23} = 3D$, the directivity pattern can be written

$$V_{out} \propto \left\{\left[\frac{\sin BX}{BX}\frac{\sin 3BX}{3BX}\right]\cos 2(2f_0X) + \left[\frac{\sin 4BX}{4BX} + \frac{\sin BX}{BX}\ \frac{\sin 3BX}{3BX}\right]\cos 4(2f_0X)\right\}$$

where $X = (\pi D/c)\sin\phi$.

The variance of the output voltage is a measure of the unwanted ac power which must be reduced by time averaging, so this quantity must be known to complete the description of a correlation array. Just as in the calculation of the average output voltage, the variance can be determined by the characteristic function method, in this case by first finding $E\left\{(V_1 V_2^2 V_3)^2\right\}$. This is the coefficient of

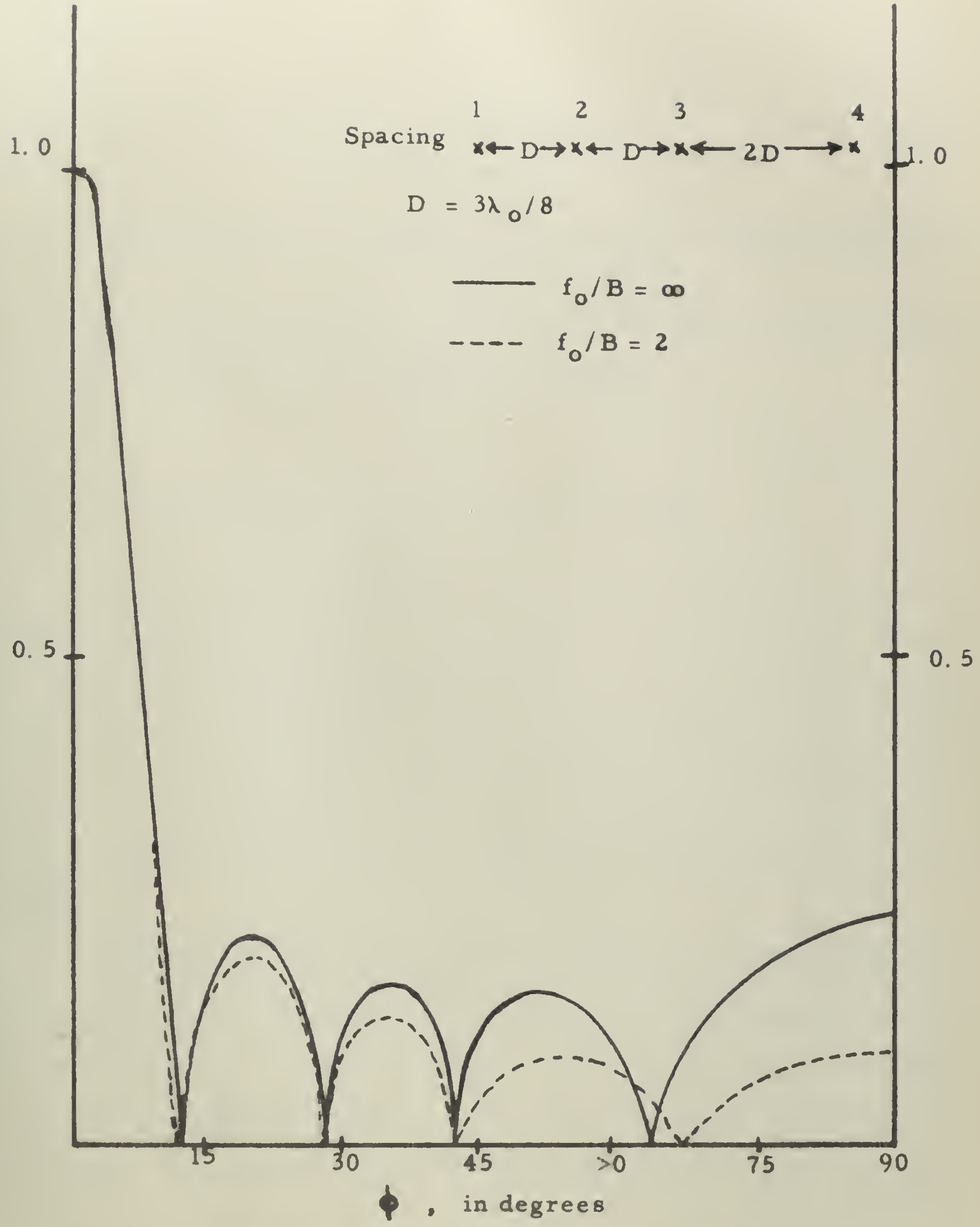


FIG. 1 -- Normalized directivity pattern, four-element correlation array.

$[\,(j\xi_1)^2/2. \cdot (j\xi_2)^4/4. \cdot (j\xi_3)^2/2.\,]$ in the same power series expansion.

$$E\left\{(V_1V_2^2V_3)^2\right\} \propto \Big[3\,R(0)^4 + 6\,R(0)^2R(\tau_{13})^2 + 12R(0)^2R(\tau_{12})^2 + 12R(0)^2R(\tau_{23})^2 + 24R(\tau_{12})^2R(\tau_{23})^2 + 48R(0)R(\tau_{12})R(\tau_{23})R(\tau_{13})\Big].$$

The mean and the variance of this array voltage, each normalized to unity at $(2f_0X) = 0^\circ$, are shown in Fig. 2.

D. Band-Limited Random Signals, Special Cases

In the previous case no frequency restriction was imposed on the multiplication circuitry. It is apparent that as the number of antenna elements increases this operation of multiplication will impose frequency bandwidth requirements which may be difficult to meet. For each stage of multiplication added to the array circuitry, frequency requirements are doubled.

As shown in the equations for the two element product array, the output voltage consists of a low frequency component and a component centered at $2f_0$. If this low frequency component of each product pair is selected by a low pass filter for use in subsequent stages of multiplication, the over-all frequency bandwidth requirements of the correlation circuits are kept within reasonable bounds.

In this case the directivity pattern cannot be obtained

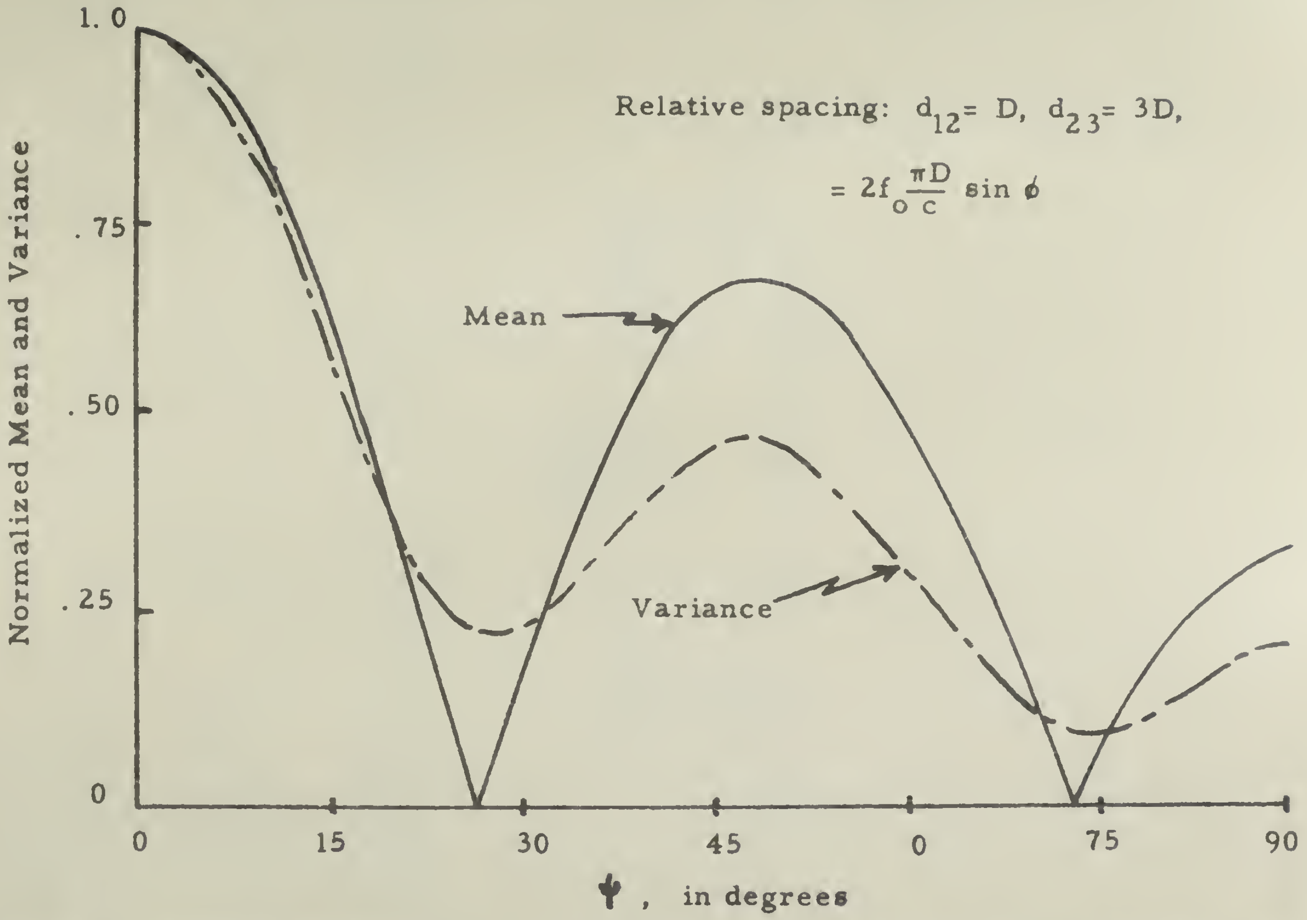
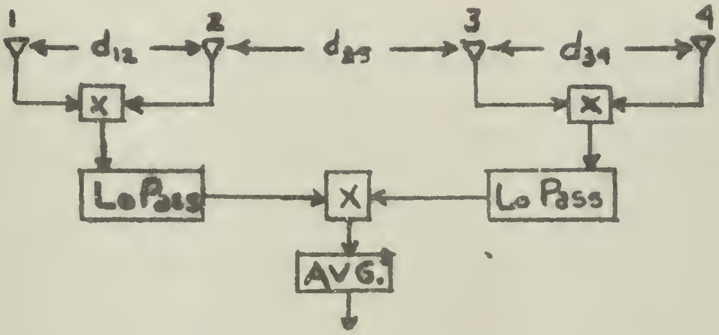


FIG. 2. -- Mean and variance of output voltage, three-element correlation array.

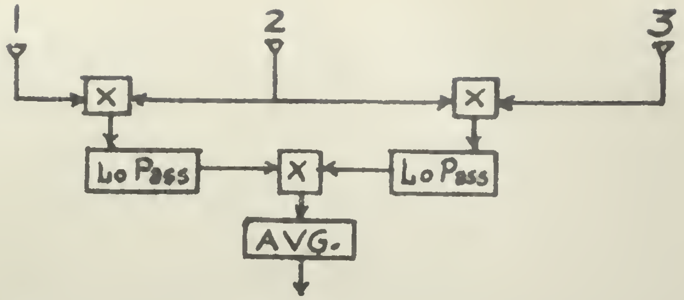
from the characteristic function but must be calculated from the low frequency term of the equation for the two-elemen product pair. By rejecting the double frequency component some increase in the width of the directivity pattern must be expected. It is also noted that the low frequency component is sensitive primarily to the spacing between the two elements of each pair. The spacing between different product pairs in the array influences the sidelobe level through the (sin nBx)/(nBx) coefficients. For example, consider the two-product-pair correlation array



$$\overline{V_{out}} \propto \left\{ \left[\frac{\sin X_{12}B}{X_{12}B} \cdot \frac{\sin X_{34}B}{X_{34}B} + \frac{\sin X_{14}B}{X_{14}B} \cdot \frac{\sin X_{23}B}{X_{23}B} \right] \cos\,(X_{12}+X_{34})2f_0 + \right.$$

$$\left. \left[\frac{\sin X_{12}B}{X_{12}B} \cdot \frac{\sin X_{34}B}{X_{34}B} + \frac{\sin X_{13}B}{X_{14}B} \cdot \frac{\sin X_{24}B}{X_{23}B} \right] \cos\,(X_{12}-X_{34})2f_0 \right\}.$$

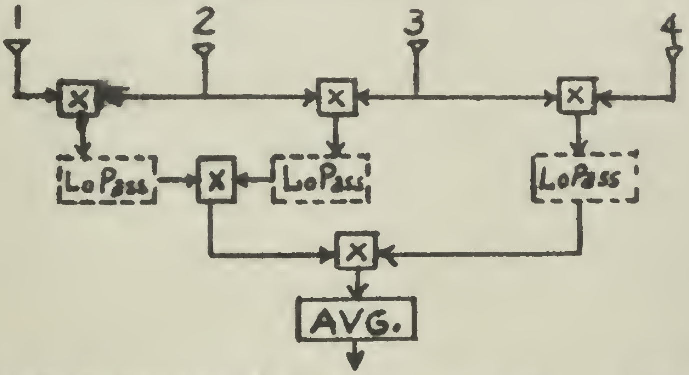
with $X_{ij} = (\pi d_{ij}/c) \sin \phi$. Since spacing between the pairs has only a secondary effect on the directivity pattern, this can be eliminated, giving two product pairs from three elements. The directivity pattern



is then given by (with relative spacing D and 3D):

$$\overline{V_{out}} \propto \left\{ \left[2\frac{\sin XB}{XB}\,\frac{\sin 3XB}{3XB} \right] \cos 2(2f_0X) + \left[\frac{\sin XB}{XB} \cdot \frac{\sin 3XB}{3XB} + \frac{\sin 4XB}{4XB} \right] \cos 4(2f_0X) \right\} .$$

As a comparison of the directivity patterns which result from the general correlation array utilizing all frequencies and from the low frequency correlation array, consider a four-element array under both conditions:



With relative spacings $d_{12} = 4D$, $d_{23} = D$, $d_{34} = 2D$, the equations for the two cases are

1. General case

$$\overline{V_{out}} \propto [A \cos 9(2f_0X) + 4B \cos 7(2f_0X) + C \cos 5(2f_0X) + 2D \cos 3(2f_0X) + 2E \cos (2f_0X)] .$$

2. Low frequency case

$$\overline{V_{out}} \propto [F \cos 7(2f_0X) + G \cos 5(2f_0X) + H \cos 3(2f_0X) + I \cos (2f_0X)]$$

In each case the coefficients A, B, ... are functions of (sin n XB)/(n XB) whose values are approximately unity for narrow band situations and become less than one only in wide-band circuits. Assuming a value of unity for these coefficients, the directivity patterns for these two cases are shown in Fig. 3.

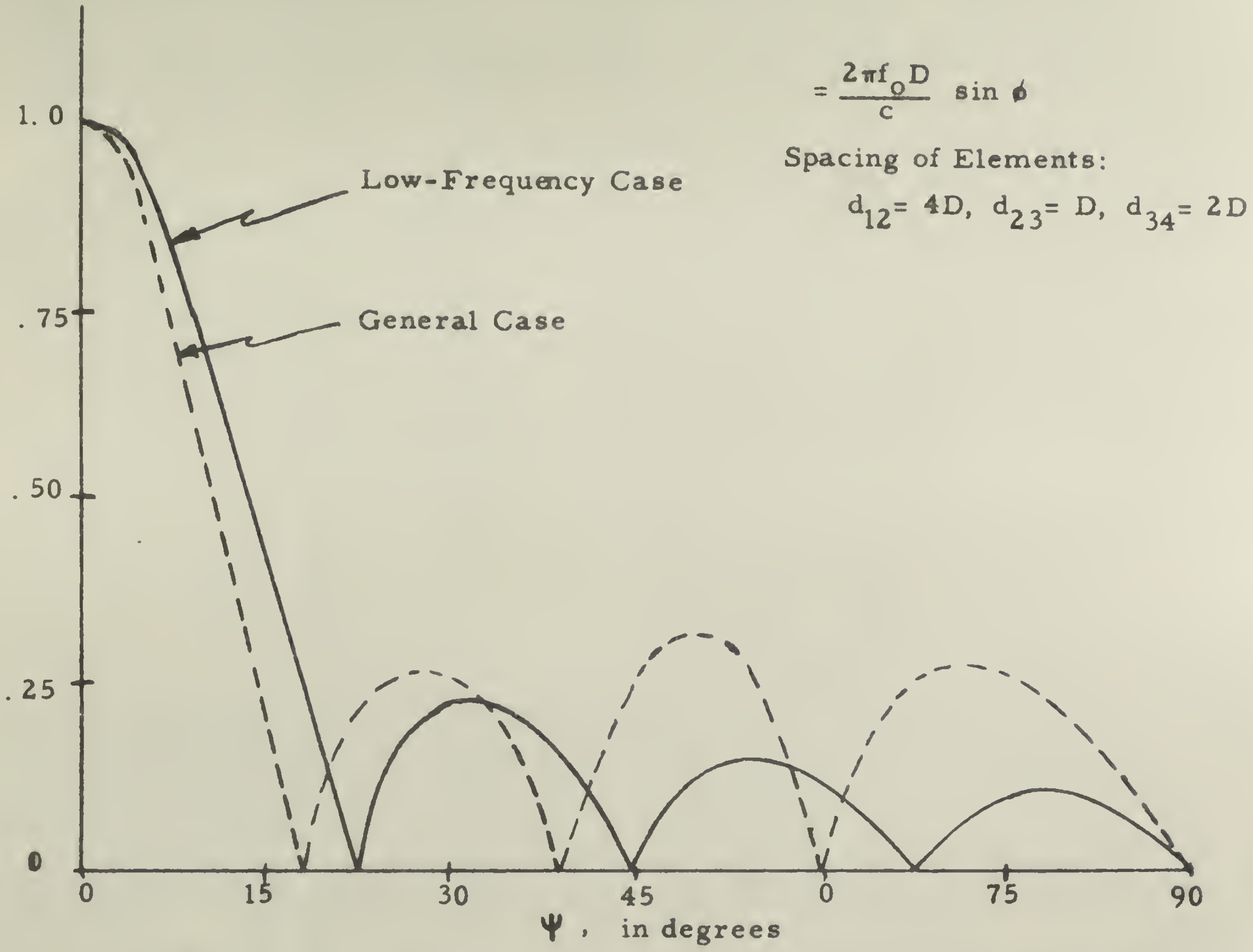


FIG. 3. -- Directivity patterns, four-element correlation arrays.

III. EFFECT OF FINITE AVERAGING TIME

The antenna directivity patterns in correlation arrays are exact only in the limit of infinite averaging time. In actual correlators this integration time would be a finite interval chosen as a compromise between the desired response of the system and the maximum time delay which can be tolerated. It becomes important, then, to investigate the system performance as a function of this integration time.

An expression for the effect of an averaging device can be obtained from general filter considerations.[9] If h(t) is the effective weighting function of a linear measuring device, and x(t) is the function to be measured, a measurement $M_x(T)$ made at time t = T after x(t) has been introduced at t = 0 can be expressed by the convolution

$$M_x(T) = \int_0^T h(u)x(T-u)\,du \qquad h(u) = 0,\ u < 0$$

where (0,T) is the observation interval. $M_x(T)$ will vary from observation to observation, fluctuating about the expected value $\overline{M_x(T)}$ with a variance $\sigma^2 = \overline{M_x(T)^2} - [\overline{M_x(T)}]^2$.

In the general situation, x(t) is at least wide sense stationary, and

$$\overline{M_x(T)} = \int_0^T h(u)\,\overline{x(T-u)}\,du$$

$$\overline{M_x(T)^2} = \int_0^T\int_0^T h(u)\,\overline{x(T-u)x(T-v)}\,h(v)\,du\,dv$$

$$= \int_0^T\int_0^T h\,(u)\,\psi_x(u-v)\,h(v)\,du\,dv$$

where ψ_x is the correlation coefficient of the function to be measured. For an ideal integrator $h(u) = 1/T$, $0 < u < T$. And

$$\overline{M_x(T)} = \overline{x(t)}$$

$$\overline{M_x(T)^2} = \frac{2}{T}\int_0^T (1 - \frac{s}{T})\,\psi_x(s)\,ds$$

This result was obtained by Davenport[10] in an early study of correlation detectors. However, his analysis was directed to a description of signal-to-noise ratio in these detectors, while the present investigation is concerned with placing confidence intervals around the expected value of the output voltage, so the methods of utilizing the information in these equations will differ.

The information required for this statistical description of the output voltage will be the expected value and the autocorrelation coefficient of the input voltage to the time averager. The calculations will be discussed in some detail for the two-element array, and then the effect of more elements in the correlation array will be indicated.

If we consider the usual two-element array, an arbitrarily located signal source induces a band-limited random signal $V_1(t)$ on element #1 and $V_2(t) = V_1(t + \tau)$ on element #2. Then, after multiplication

$$Y(t) = V_1(t)V_2(t) = V_1(t)V_1(t + \tau)$$

$$\overline{Y(t)} = R_V(\tau)$$

$$\begin{aligned}\Psi_V(s) &= \overline{Y(t)Y(t+s)} = \overline{V_1(t)V_1(t+\tau)V_1(t+s)V_1(t+\tau+s)} \\ &= R_V(\tau)^2 + R_V(s)^2 + R_V(\tau + s)R_V(\tau - s)\end{aligned}$$

The last term of this expression for the autocorrelation coefficient of the product voltage becomes rather awkward to handle in the equations for the integrator circuit. And as more elements are added to the array, the corresponding terms in their product equations increase the complexity of the calculations. Since the interest here is in the change in integration time required as the number of elements in the correlation array increases, an approximation for the corelation coefficient will be used which will permit straightforward calculation of the integration characteristics.

If the signal source is located in the principal lobe ($\tau = 0$), the equations for the product voltage become

$$\overline{Y(t)} = R_V(0)$$

$$\begin{aligned}\Psi_V(s) &= R(0)^2 + 2R(s)^2 \\ &= R(0)^2(1 + 2\rho(s)^2)\end{aligned}$$

where $\rho(s)$ is the normalized autocorrelation coefficient; $\rho(s)=R(s)/R(0)$. Then for an arbitrarily located source, the approximate product voltage equations are

$$\overline{Y(t)} = R_V(\tau)$$

$$\Psi_V(s) \approx R_V(\tau)^2(1 + 2\rho(s)^2)$$

This approximation is good for positions near the principal lobe axis and fails for positions that make $R_V(\tau)$ quite small ($R_V(\tau) \rightarrow 0$).

For the usual rectangular band-limited voltage, $\rho(s) = (\sin \pi Bs/\pi Bs) \cos 2\pi f_0 s$, and the equations for the ideal integrator are

$$\overline{M_Y(T)} = R_V(\tau)$$

$$\overline{M_Y(T)^2} = R_V(\tau)^2 \frac{2}{T} \int_0^T (1 - \frac{s}{T})\left(1+2\frac{\sin^2\pi Bs}{(\pi Bs)^2} \cos^2 2\pi f_0 s\right) ds$$

$$\overline{M_Y(T)^2} = R_V(\tau)^2 \frac{2}{T} \int_0^T (1 - \frac{s}{T})[1+\left(\frac{\sin\pi Bs}{\pi Bs}\right)^2 +$$

$$\left(\frac{\sin\pi Bs}{\pi Bs}\right)^2 \cos 2\pi(2f_0)s]ds$$

The solution of this integral can be expressed in two parts, the first depending on the low frequency components of the product voltage, the second determined by the double frequency component.

$$M_Y(T)^2 = R_V(\tau)^2[1 + m_{1f}^2 + m_{2f}^2]$$

Or, in terms of the variance at the output of the integrating circuit,

$$\sigma^2 = R_V(\tau)^2\ [m_{1f}^2 + m_{2f}^2]$$

where

$$m_{1f}^2 = \left[- \frac{1 - \cos 2\pi BT}{(\pi BT)^2} + \frac{2}{\pi BT}\mathrm{Si}(2\pi BT) - \frac{1}{(\pi BT)^2}\mathrm{Cin}(2\pi BT)\right]$$

$$\approx \frac{1}{BT}\text{, for } BT \gg 1.$$

$$m_{2f}^2 = \left[\frac{\mathrm{Si}(2\pi BT)}{\pi BT} - \frac{\mathrm{Si}(2k\pi BT)}{\pi BT} + \frac{(k+1)\mathrm{Si}(2k+2)\pi BT}{2\pi BT} + \frac{(k-1)\mathrm{Si}(2k-2)\pi BT}{2\pi BT} + \frac{\sin^2\pi BT}{(\pi BT)^2}\cos^2 k\pi BT - \frac{\mathrm{Cin}(2\pi BT)}{2(\pi BT)^2} - \frac{\mathrm{Cin}(2k\pi BT)}{2(\pi BT)^2} + \frac{\mathrm{Cin}(2k+s)\pi BT}{4(\pi BT)^2} + \frac{\mathrm{Cin}(2k-s)\pi BT}{4(\pi BT)^2}\right]$$

where $k = (4f_0)/B$, $\mathrm{Si}(x)$ is the sine integral, and $\mathrm{Cin}(x)= \int_0^x (1-\cos v/v)\,dv$.

The standard deviation of the voltage at the output of the integration circuit as a function of the integration time, T, is shown in Fig. 4. In the limit as $T \rightarrow 0$, the variance is $2\,R_V(\tau)^2 = \overline{2M_Y(T)}^2$, as would be expected for the Chi-square distribution. For voltage samples observed without integration, a 95 per cent confidence interval on the output voltage extends from zero to 3.87 $\overline{M_Y(T)}$. As

integration time increases, the confidence interval becomes smaller, and observed voltage samples would cluster more and more closely about the expected value, $\overline{M_Y(T)}$. Figure 5 shows this narrowing of the 95 per cent confidence interval as integration time increases.

It should be noted that the double frequency component of variance decays rapidly compared to the low frequency component. A correlator whose integration time is long compared to $1/f_0$ may still have an output voltage with an appreciable variation because of this low frequency variance. If the input circuits of the integrator rejected this double frequency component of the product voltage, the output voltage would fluctuate with only the low frequency component of variance. Regardless of this circuitry, however, the integration time that will be necessary to reduce the fluctuation will be very long compared to $1/f_0$, and the double frequency component of variance can generally be completely neglected.

Incidentally, this statistical description of the fluctuation of the output voltage of a two-element correlation array is the same as the statistical description of the signal power output of a linear additive array. The only difference would occur in the magnitude of the expected value, $\overline{M_Y(T)}$. Knowing the variance of the linear array power fluctuation as a function of time, we can compare averaging time requirements of more complex correlation arrays with that of the familiar additive array.

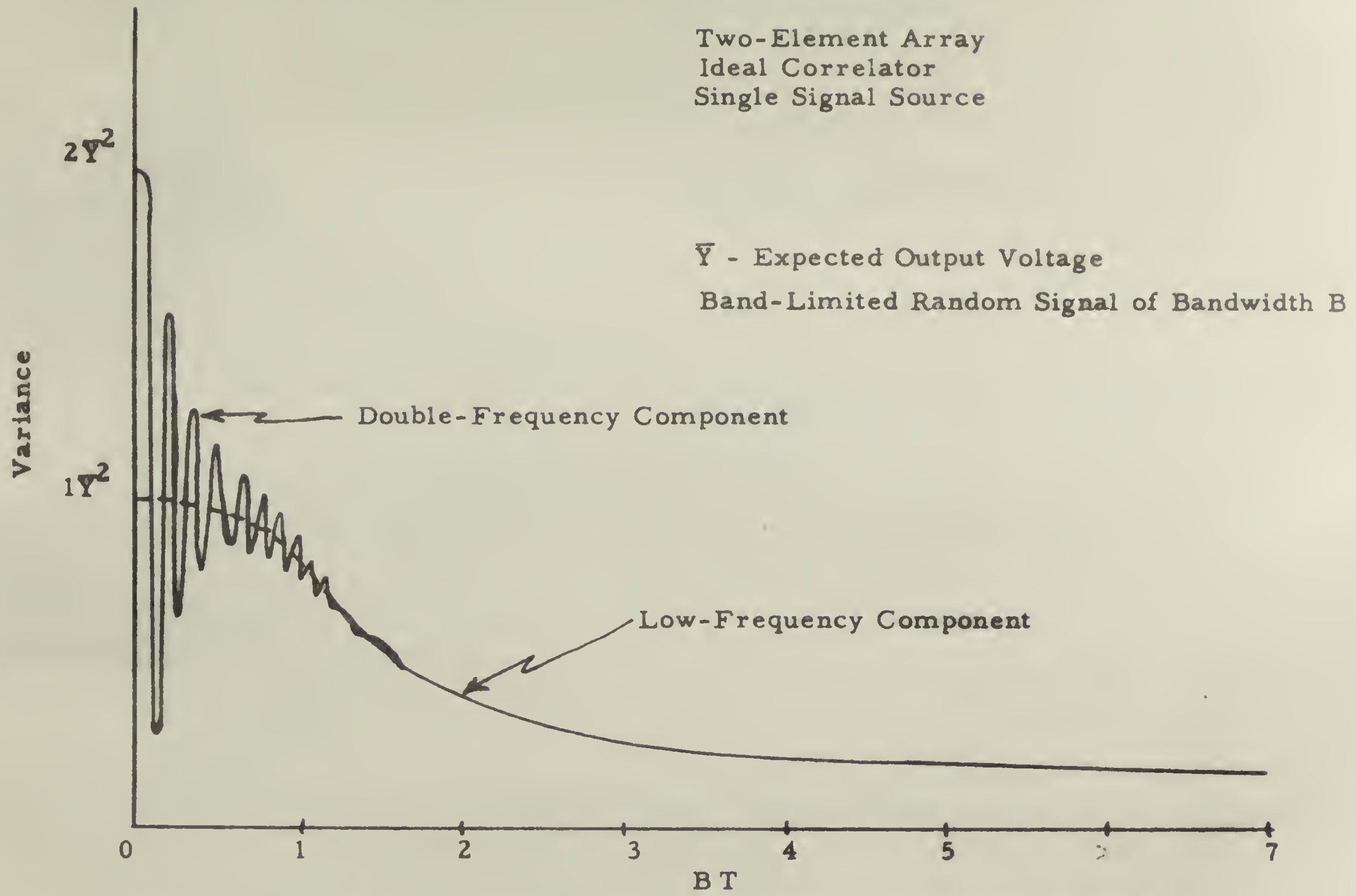


FIG. 4. -- Variance of output voltage as a function of integration time, T.

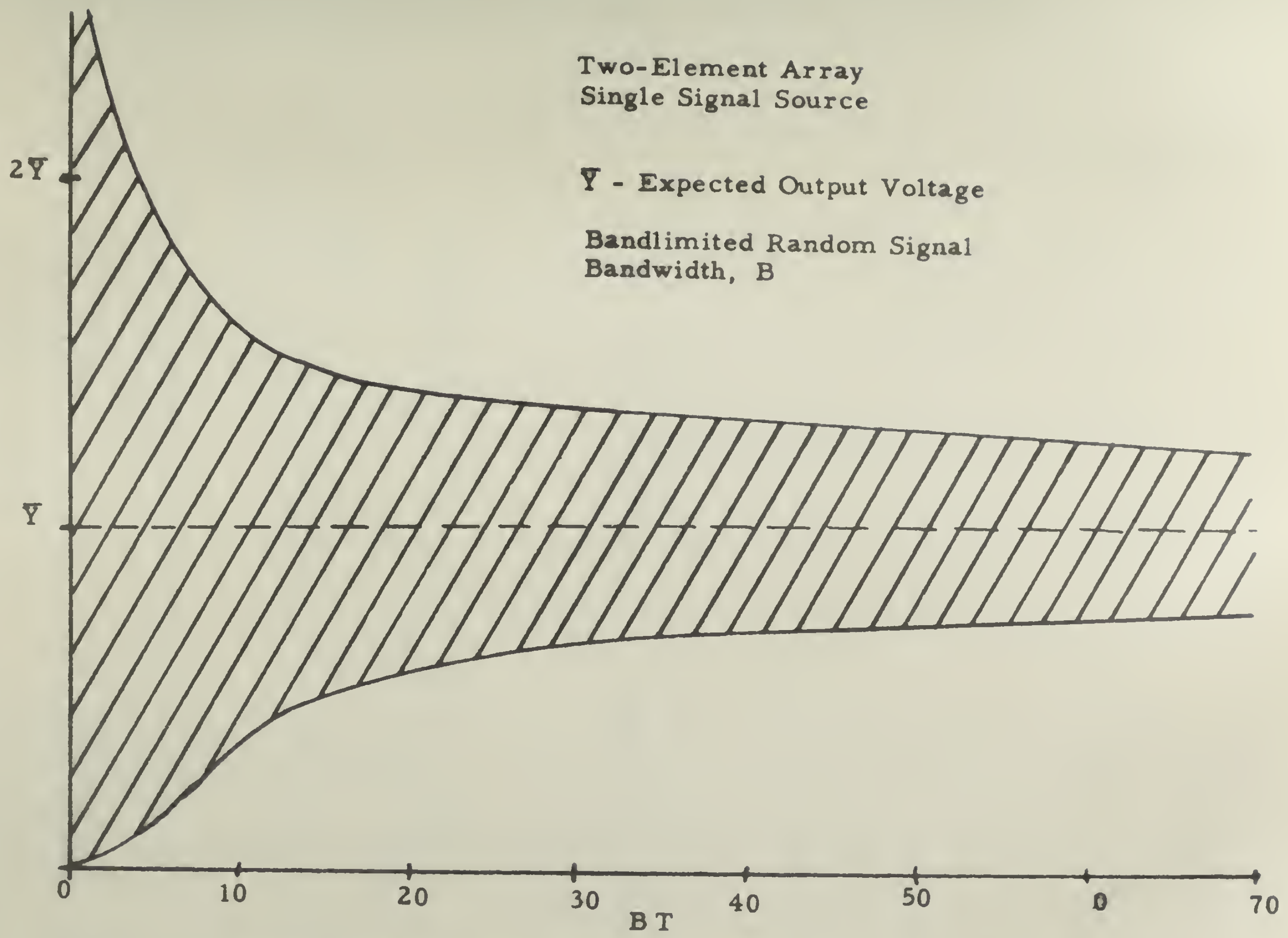


FIG. 5. -- Ninety-five per cent confidence interval as a function of integration time.

The autocorrelation coefficient of the product voltage, Y(t), for general correlation arrays will be calculated in the same manner as for the two-element array. The expected value of the product voltage can be determined from the charcteristic function as in the preceding section. The autocorrelation coefficient of the product voltage for a source located on the main lobe axis can also be determined by the charcteristic function method. This will be modified by inserting the expected value of the arbitrarily located source, and the resulting form will be used to approximate the actual autocorrelation coefficient.

The expected value of product voltage is given in the following table for a few values of the number of elements, n.

n	$\overline{Y(t)}$
2	$R(\tau_{12})$
4	$R(\tau_{12})R(\tau_{34})+R(\tau_{13})R(\tau_{24})+R(\tau_{14})R(\tau_{23})$
6	$\sum_{m,n,o,p,q=2}^{6} R(\tau_{1m})R(\tau_{no})R(\tau_{pq}),\ m \neq n \neq o \neq p \neq q$

The characteristic function for the product voltage is

$$F(\xi_1, \xi_2, s) = E\left\{\exp\ [j\xi_1 V(t) + j\xi_2 V(t+s)]\right\}$$

$$= \exp\left[-\frac{R(0)}{2}\ (\xi_1^{\ 2} + \xi_2^{\ 2}) - R(s)\,\xi_1 \xi_2\right]$$

Then $\Psi_n(s)$ is the coefficient of $(j\xi_1)^n/n \cdot (j\xi_2)^n/n.$ in

the power series expansion of $F(\xi_1, \xi_2, s)$.

n	$\psi_n(s)$
2	$\overline{Y(t)}^2\ [1 + 2\rho(s)^2]$
4	$\overline{Y(t)}^2\ [9 + 72\rho(s)^2 + 24\rho(s)^4]$
6	$\overline{Y(t)}^2\ (225 + 4050\rho(s)^2 + 5400\rho(s)^4 + 720\rho(s)^6]$

Calculation of the variance, for the usual rectangular band-limited voltage, gives

for two elements (and for signal power in linear array)	$\sigma_Y^2 = \overline{Y(t)^2}\ \frac{1}{BT}$
for four elements	$\sigma_Y^2 = \overline{Y(t)^2}\ \frac{4.68}{BT}$
for six elements	$\sigma_Y^2 = \overline{Y(t)^2}\ \frac{15.52}{BT}$

in all cases, $BT >> 1$

This last table shows the general relationship between variance and number of elements in a correlation array. For any established level of fluctuation in the output voltage, a six-element correlation array will require an integration time 15.5 times as long as is required for a linear array (or a two-element correlation array). In the previous section the directivity pattern of a correlation array was found to be similar to that of a much longer linear array. Thus the correlation process in antennas is seen to be essentially one of trading space for time.

IV. RESOLUTION CHARACTERISTICS OF CORRELATION ARRAYS

In describing the capability of a correlation array to resolve two similar signal sources separated by an angular displacement θ, the directivity patterns developed in Section III cannot be applied directly. Since the correlation arrays depend on multiplicative operations, the presence of two sources causes cross-product terms to appear. These terms then necessitate additional calculations to determine resolution capability. These calculations will be examined for correlation arrays in the presence of single frequency sources and randomly varying sources. The correlation arrays will be found to have resolution capabilities equivalent to those of longer linear arrays, if the sources to be resolved are independent; however, for coherent sources resolution may be possible only for certain specific separations of the sources and not as a general rule.

First, let us examine a four-element correlation array as developed in Section III. Assume there are two monochromatic sources, source A ($V_A = A \cos \omega_A t$) and source B ($V_B = \cos \omega_B t$), separated by an angular displacement ϕ°. The general expression for the voltage response of the array as a function of position of the sources is quite lengthy, and is developed in Appendix II. It is found to depend on the coherence of the sources. The two cases that occur can be summarized:

1. For two coherent sources:

$$V_{out} \propto A^4(\cos X + \cos 3X + \cos 5X) + B^4(\cos Y + \cos 3Y + \cos 5Y)$$
$$+A^3B\ f[\cos(nX-mY)] + A^2B^2f[\cos(nX-mY)] + AB^3f[\cos(nX-mY)]$$

2. For two sources of slightly different frequencies:

$$V_{out} \propto A^4(\cos X + \cos 3X + \cos 5X) + B^4(\cos Y + \cos 3Y + \cos 5Y)$$
$$+A^3B\ f[\cos(nX-mY)\cdot \cos\delta t] + AB^3f[\cos(nX-mY)\cdot\cos\delta t]$$
$$+A^2B^2f[\cos(nX-mY)\cdot\cos\delta t] + A^2B^2f[\cos(nX-mY)]$$

where $X = (\omega D/c)\sin\phi_A$ and $Y = (\omega D/c)\sin\phi_B$ are the phase delay factors for the two sources with element spacing D, and $f[\cos(nX-mY)]$ represents a number of cosine terms containing these phase delay factors, while $f[\cos(nX-mY)\cdot\cos\delta t]$ represents a number of these cosine terms modified by the beat frequency component, δt, between source A and source B.

In both of these equations, the first two terms correspond to the voltage response pattern of a linear antenna array in the presence of two sources. The remaining terms arise from the nonlinearity of the correlation array. In the case of two sources with identical frequencies, the cross-product terms are constant with time and averaging the output voltage will not alter the result. When the sources have different frequencies, however, time averaging can be employed to reduce the beat frequency ($\cos\delta t$) part of the cross-product terms; but even in this case the resulting voltage response pattern will not be precisely equivalent to that of the linear array, since

some of the cross-product terms will remain constant with time and will contribute to the dc component of the output voltage.

The voltage response pattern of this four-element correlation array as the array is rotated past a pair of coherent signal sources separated by an angular displacement of 20° is shown in Fig. 6. In this case the cross-products result in an apparent indication of a single source located midway between the actual pair of sources. This four-element correlation array will resolve sources with an angular separation of 14° and with an angular separation of 28°, but a pair of sources whose angular separation is in the range 0°-13° or in the range 15°-26° will not be resolved by the array.

It should be noted that if the correlator contains only one stage of multiplication, then the cross-product terms occurring from two sources of different frequencies appear only as beat frequency components. These terms describe a low frequency ac signal appearing with the desired dc measurement voltage. In this case the cross-product terms can be minimized by time averaging the output voltage. However, if more than one stage of multiplication occurs between the antenna element and the output of the array (as is the case in the example of the four-element correlation array), the cross-product terms will occur as low frequency ac terms and also as dc terms. In this case the equivalence between the correlation arrays and the linear arrays is not immediately apparent but must be determined by calculation of resolution characteristics for the

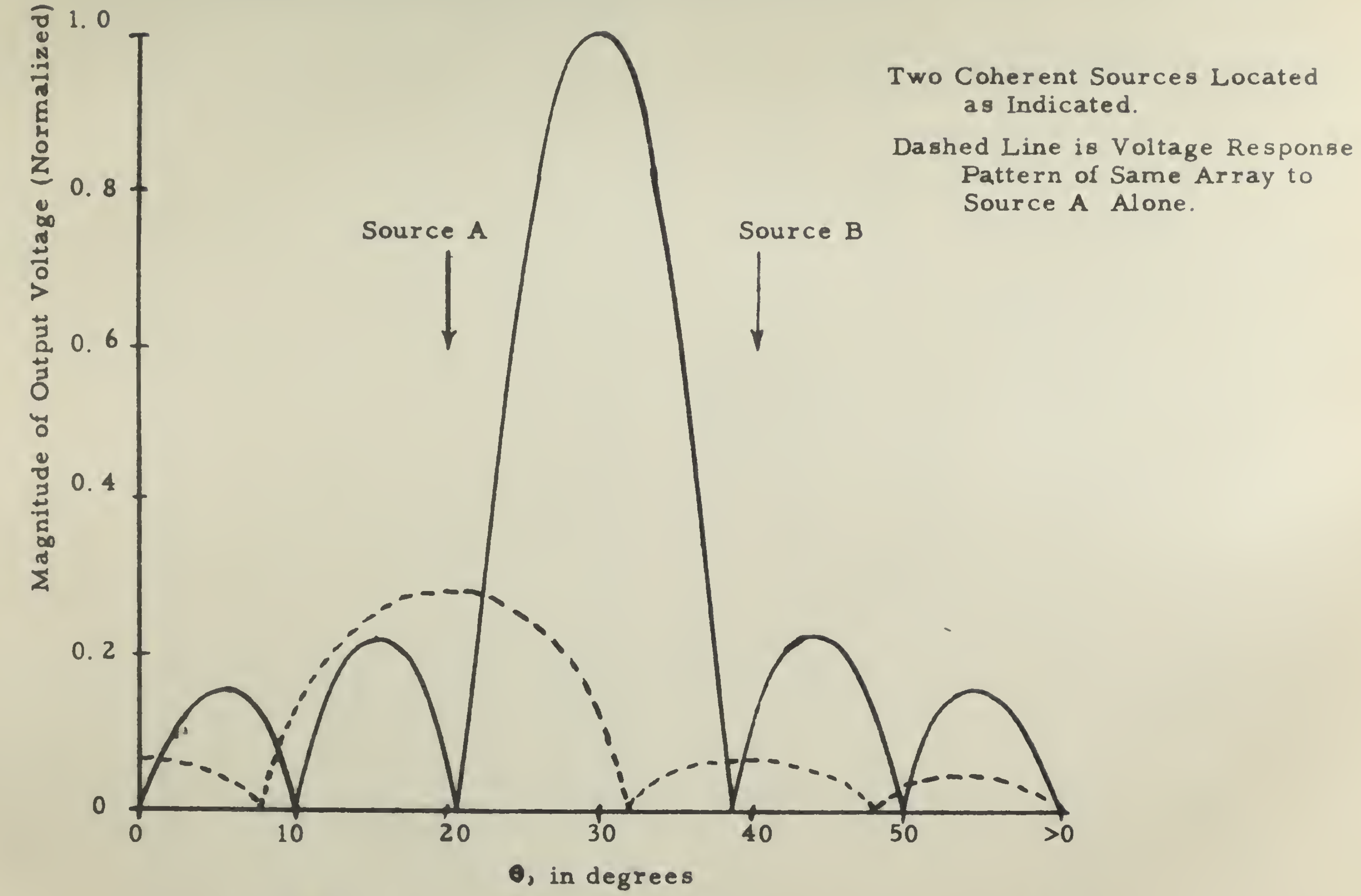
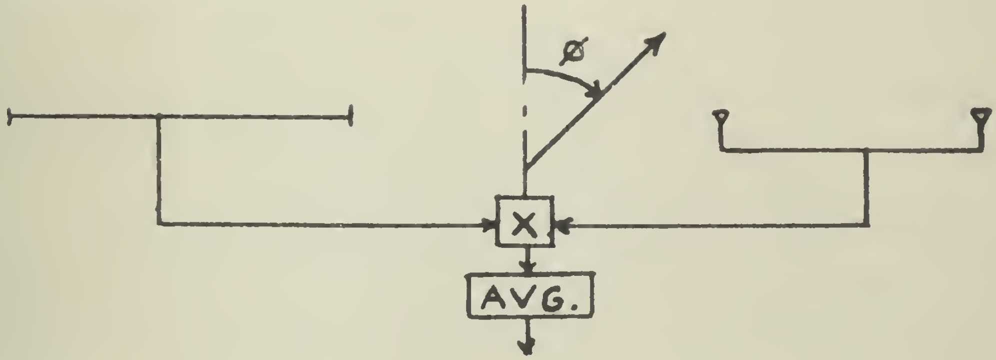


FIG. 6. -- Voltage response pattern, four-element correlation array.

particular type of signal encountered.

Assuming a basic spacing D = $\lambda/2$, the four-element correlation array of length 4D will resolve two sources of slightly different frequencies at approximately 19.5°. This is equivalent to the resolution of a uniform linear array with an aperture of 9D, about twice as long.

A slightly more complex example of a correlation array is one discussed by Drane:[5]



The element on the left represents a uniform linear array with aperture 2a, while that on the right is a simple interferometer with aperture 2b.

The directivity pattern of the uniform linear array is proportional to (sin X)/X, where $X = \omega(a/c)\sin\phi$. The directivity pattern of the interferometer is proportional to cos Y, where $Y = \omega(b/c)\sin\phi$. If a=b and there is no spacing between the right end of the linear array and the left element of the interferometer, the directivity pattern of the correlation array is

$$V_{out} \propto \frac{\sin 4X}{4X}$$

which is the same as the directivity pattern of a uniform linear

array of twice the length of this correlation array.

However, if two coherent sources are present the output becomes

$$V_{out} \propto \frac{\sin 4X_A}{4X_A} + \frac{\sin 4X_B}{4X_B} + \cos (X_A + X_B) \frac{\sin X_A \cos X_B}{4X_A} + \frac{\cos X_A \sin X_B}{4X_B}$$

The third term again occurs because of the nonlinearity of the correlator. Calculation of the resolution capability of this array shows that it is equivalent to a uniform linear array 1-1/2 times as long as the correlation array.

These have been only two examples of the effect of cross-product terms in a correlation array in the presence of monochromatic sources. In each case that might be considered, the effect of these terms would vary, depending on the types of signal voltages emitted by the sources and by the number of successive multiplicative processes in the correlator.

When two statistically similar sources are inducing randomly varying voltages on the antenna elements, the resolution capabilities of the correlation array must be expressed in terms of integration times and confidence intervals. Consider again the four-element correlation array and two independent sources (source A and source B). Then the expected value of the output voltage can be calculated either directly or by the characteristic function method and can be expressed in terms of the correlation coefficients of the individual

voltages. The voltage induced on the i-th element is

$$V_i(t) = V_A(t + \tau_i) + V_B(\tau + \tau_i)$$

and

$$\begin{aligned}\overline{Y(t)} = \overline{V_1(t)V_2(t)V_3(t)V_4(t)} = \\ &[R_A(\tau_{12})R_A(\tau_{34}) + R_A(\tau_{13})R_A(\tau_{24}) + R_A(\tau_{14})R_A(\tau_{23})] \\ &+ [R_B(\tau_{12})R_B(\tau_{34}) + R_B(\tau_{13})R_B(\tau_{24}) + R_B(\tau_{14})R_B(\tau_{23})] \\ &+ [R_A(\tau_{12})R_B(\tau_{34}) + R_A(\tau_{13})R_B(\tau_{24}) + R_A(\tau_{14})R_B(\tau_{23}) \\ &+ R_A(\tau_{34})R_B(\tau_{12}) + R_A(\tau_{24})R_A(\tau_{13}) + R_A(\tau_{23})R_B(\tau_{14})\end{aligned}$$

The first and second lines in this expression give the expected voltages due to each source individually. The third and fourth lines contain the cross-product components which occur as a result of the two stages of multiplication of the element voltages.

The variance of this output voltage could be calculated directly; however, the large number of terms in the final expression would make this quite laborious. Therefore, this quantity will be approximated as previously by considering the variance of the voltage produced by a single source on the main lobe axis and then introducing the new value of $\overline{Y(t)}$ given by the equation above.

With an arbitrary angular displacement between the two sources, and a given level of confidence desired in the result,

it is possible to calculate the maximum variance which will permit a significant difference in the amplitude of the output voltage when the array is directed at a source and when it is directed between the sources. This value of variance can then be used to determine the minimum permissible integration time. So the array resolution is a function of two quantities: level of confidence and integration time.

The resolution capabilities of the simple four-element correlation array are listed in the following table. They are compared with those of a linear array.

Random Signals

Resolution at the 95 Per Cent Confidence Level

Four-Element Correlation Array Length 4D		Aperture of Equivalent Uniform Linear Array
Resolution	Integration Time	
19°	BT = 18	9.6 D
18°	BT = 97	9.4 D
17.5°	BT = 890	9.2 D
17°	BT = ∞	

V. EFFECT OF DISTRIBUTED NOISE ON THE NONLINEAR ANTENNA SYSTEM

The following assumptions will be made:

1. uniform distribution of independent random noise sources;

2. noise sources in an element $d\Omega$ induce a noise power $\eta\, d\Omega$ watts/cps in an isotropic antenna element;

3. antenna element and associated circuitry have a rectangular passband of bandwidth B, center frequency f_o, with $f_o \gg B$.

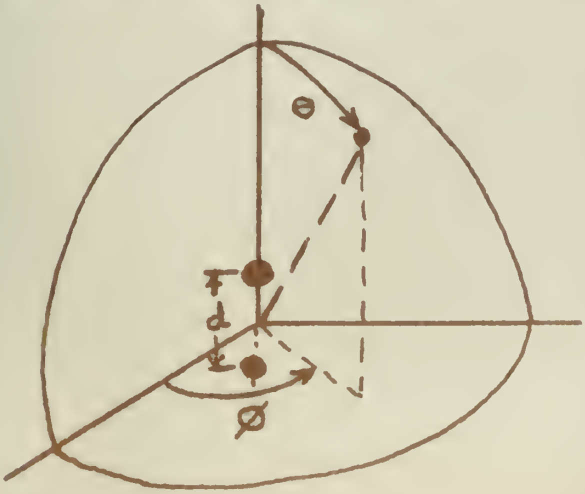
Then the autocorrelation function of the noise voltage induced on an antenna element is

$$\Psi_{11}(s) = \overline{x_1(t)x_1(t+s)} = 4\pi B\eta \frac{\sin \pi Bs}{\pi Bs} \cos \omega_o s$$

$$= R_n(0) \frac{\sin \pi Bs}{\pi Bs} \cos \omega_o s$$

where $R_n(0)$ represents the average noise power.

With two identical isotropic elements, separated by a distance d, the voltage induced on one element by the noise sources in an angular area $d\Omega$ differs from that induced on the second antenna element only by a delay time τ .



$$\tau = \frac{d}{c} \cos \theta$$

The same average power, $R_n(0)$, is induced on each element. The autocorrelation coefficient of the noise voltage on each element is the same.

The cross-correlation coefficient becomes:

$$\Psi_{12}(s) = \overline{x_1(t)x_2(t+s)} = \overline{x_1(t)x_1(t+\tau+s)}$$

$$= \int_0^{2\pi}\int_0^{\pi} B\eta \frac{\sin \pi B(\tau+s)}{\pi B(\tau+s)} \cos \omega_0(\tau+s) \sin\theta \, d\theta \, d\phi$$

$$= 2\pi B\eta \int_0^{\pi} \frac{\sin \pi B\left(\frac{d}{c}\cos\theta+s\right)}{(\pi B \frac{d}{c}\cos\theta+s)} \cos \omega_0\left(\frac{d}{c}\cos\theta+s\right) \sin\theta \, d\theta$$

This can be integrated to yield:

$$\Psi_{12}(s) = \frac{1}{4}R_n(0)\frac{c}{\pi Bd}\left\{ \text{Si}\left[(k+1)\frac{\pi Bd}{c}+\pi Bs)\right] - \text{Si}\left[(k+1)(\frac{\pi Bd}{c}+\pi Bs)\right] - \text{Si}\left[(k-1)(\frac{\pi Bd}{c}+\pi Bs)\right] + \text{Si}\left[(k-1)(\frac{-\pi Bd}{c}+\pi Bs)\right]\right\}$$

where $k = \dfrac{\omega_0}{\pi B}$ and Si(x) is the sine integral.

By the symmetry of the integrand and of the range of interpretation it is evident that

$$\Psi_{21}(s) = \overline{x_2(t)x_1(t+s)} = \overline{x_1(t+\tau)x_1(t+s)} = \Psi_{12}(s).$$

Under the usual conditions of $f_0 \gg B$, and for other than very close element spacings, this expression for the cross-correlation coefficient can be considerably simplified. Under these conditions, approximate Si(x) by $\frac{\pi}{2} - \frac{\cos x}{x}$, and, after

some manipulation, find

$$\Psi_{12}(s) = R_n(0)\frac{\sin k\frac{\pi Bd}{c}}{k\left(\frac{\pi Bd}{c}\right)}\ \frac{\sin\frac{\pi Bd}{c}}{\frac{\pi Bd}{c}}\ \frac{\sin \pi Bs}{\pi Bs}\ \cos \omega_0 s$$

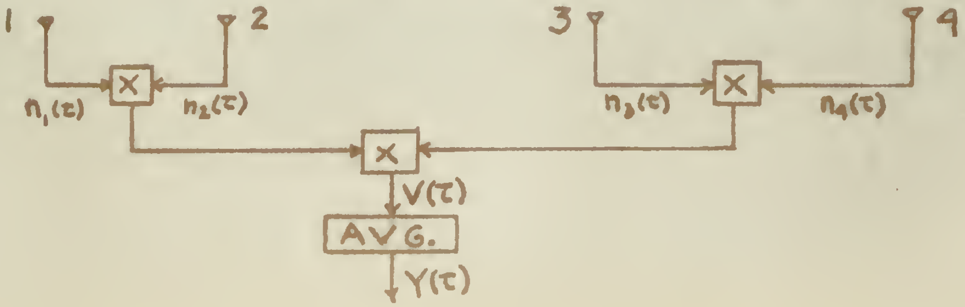
$$\approx R_n(\tau)\ \frac{\sin \pi Bs}{\pi Bs}\ \cos \omega_0 s$$

where

$$R_n(\tau) = R_n(0)\ \frac{\sin\left(\frac{\omega_0 d}{c}\right)}{\frac{\omega_0 d}{c}}\ \ \frac{\sin\left(\frac{\pi Bd}{c}\right)}{\frac{\pi Bd}{c}}$$

Since in most cases $k \gg 1$, while d, at the most is a few wavelengths, the last factor, $\sin(\frac{\pi Bd}{c})/(\frac{\pi Bd}{c})$, will have very little influence on the expression; and the cross-correlation coefficient can be considered to be a function of $\sin(\frac{\omega_0 d}{c})/(\frac{\omega_0 d}{c})$. It should be remembered that the approximation used to simplify the sine integrals in the original equation is not valid for small values of element spacing, d; and, in the event of close element spacings, calculation of the cross-correlation coefficient would require use of the exact equation.

Calculation of noise voltage and noise power present at the output of a simple nonlinear array will demonstrate the considerations which must enter into a specification of antenna element spacing. Take for example a four-element array:



The noise sources are assumed to be uniformly distributed over the surface of a unit sphere surrounding the array. Then the average value of output voltage, $\overline{Y(t)}$, becomes:

$$\overline{Y(t)} = \overline{V(t)} = \overline{n_1(t)n_2(t)n_3(t)n_4(t)}.$$

$$= R_n(\tau_{12})R_n(\tau_{34}) + R_n(\tau_{13})R_n(\tau_{24}) + R_n(\tau_{14})R_n(\tau_{23}).$$

Then, as shown in a previous section, the average value of output power is:

$$\overline{Y(t)^2} = \frac{2}{T}\int_0^T (1 - \frac{s}{T})\ \overline{V(t)V(t+s)}\ ds$$

But, with four elements,

$$\overline{V(t)V(t+s)} = \overline{n_1(t)n_2(t)n_3(t)n_4(t)n_1(t+s)n_2(t+s)n_3(t+s)n_4(t+s)}$$

And, when expressed as multiples of the correlation coefficients of the element noise voltages, this results in an equation with some 105 terms. However, if spacings are selected to make $\Psi_{ij}(s) = 0,\ (i \neq j)$, then

$$\overline{V(t)V(t+s)} = \left[\Psi_{11}(s)\right]^4 = R_n(0)^4\left(\frac{\sin \pi Bs}{\pi Bs}\right)^4 \cos^4 \omega_0 s$$

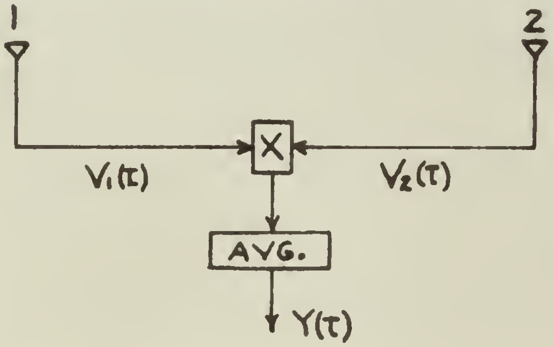
and the output power can be readily evaluated. Spacings which give this result are multiples of a half-wavelength at the center frequency, f_0.

Since the cross-correlation coefficients can assume both positive and negative values, it is conceivable that some decrease

in output noise power could be realized by appropriate changes in element spacings after a study of the complete expression for $\overline{V(t)V(t+s)}$ under the existing conditions of noise source distribution. However, the above method of considering all cross-correlation coefficients as zero gives a quick evaluation of output power which, by the very nature of the sin x/x function, is subject only to minor modification by more detailed considerations.

This development of the output noise power from a nonlinear array does not give the complete description of the operation of this type of array when it is used to detect a signal buried in a general noisy medium. Because of the multiplicative operations performed in the antenna circuitry, there will be an interaction between signal and noise which can appear as an undesired component of the output voltage.

This interaction will be shown by considering a simple two-element correlation array. Assume a single signal source induces a voltage which is in phase on each element, and a uniform spherical distribution of noise sources induces undesired noise voltages on the elements.



Then

$$V_1(t) = S(t) + n_1(t)$$

$$V_2(t) = S(t) + n_2(t) = S(t) + n_1(t+\tau_{12}).$$

$$\overline{Y(t)} = \overline{S^2(t)} + \overline{S(t)\ n_1(t)+\ n_2(t)} + \overline{n_1(t)n_2(t)}$$

$$= \overline{S^2(t)} + R_n(\tau_{12}) = \overline{S^2(t)} + \Psi_{12}(0)$$

If the element spacing is chosen (multiple of a half wavelength) to make $\Psi_{12}(0) = 0$, then the voltage in the output circuit due to the signal is

$$\overline{Y(t)} = \overline{S^2(t)}.$$

Noise power, N, in the output circuit, can be determined from

$$N = \frac{2}{T}\int_0^T (1 - \frac{s}{T}) \left[\overline{Y(t)Y(t+s)} = \overline{Y(t)}^2\right] ds\ .$$

$$= \frac{2}{T}\int_0^T (1 - \frac{s}{T})\left\{\overline{S^2(t)S^2(t+s)} + \overline{S(t)S(t+s)}\left[\overline{n_1(t)n_1(t+s)} + \overline{n_2(t)n_2(t+s)}\right] + \overline{n_1(t)n_2(t)n_1(t+s)n_2(t+s)} - \overline{S^2(t)S^2(t)}\right\} ds$$

retaining only terms whose expectations are nonzero.

If the signal voltage induced on a single element is a simple sinusoid, $S = A \cos \omega t$, then the signal voltage in the output circuit is

$$\overline{Y(t)} = \tfrac{1}{2}A^2.$$

The signal power in the output circuit is

$$P = \tfrac{1}{4}A^4,$$

and the noise power is

$$N = \frac{2}{T}\int_0^T \left(1 - \frac{s}{T}\right)\left\{\left[A^2 \cos \omega s\right]\left[R_n(0)\frac{\sin \pi Bs}{\pi Bs}\cos \omega s\right] + R_n(0)^2\left[\frac{\sin \pi Bs}{\pi Bs} \cos \omega s\right]^2\right\} ds.$$

Since $f_o >> B$, the ac component of the noise will be essentially zero in averaging times such that $BT > 1$. So only the low frequency component of the noise power in the above equation need be considered.

$$N = \frac{2}{T}\int_0^T \left(1 - \frac{s}{T}\right)\left\{\frac{A^2 R_n(0)}{2} \frac{\sin \pi Bs}{\pi Bs} + \frac{R_n(0)^2}{2} \frac{\sin^2 \pi Bs}{(\pi Bs)^2}\right\} ds$$

For large values of BT, this can be approximated by

$$N \approx \frac{A^2 R_n(0)}{2} \frac{1}{BT} + \frac{R_n(0)^2}{2} \frac{1}{BT}$$

So the noise power appears not only as a result of the noise sources alone (last term) but also as a result of the interaction or signal and noise voltages (middle term).

This total noise power resulting from the interaction of signal and noise voltages may well be greater than that caused by the noise sources alone. For this two-element nonlinear array and under conditions of unity signal-to-noise ratio on an individual element, the total noise power resulting from the cross multiplication is twice the noise power caused by the noise sources alone.

Nonlinear arrays of this same type (combining network

composed of product circuits only), but with greater numbers of elements, will have additional noise power terms resulting from this cross multiplication of signal and noise voltages. This is summarized in the following table. In this table, the level of multiplication refers to the number of times the voltage induced on an element undergoes multiplication before reaching the output circuit.

SIGNAL POWER AND NOISE POWER IN THE OUTPUT CIRCUIT OF A NONLINEAR ANTENNA ARRAY

Signal Voltage induced on each element $S = A \cos \quad t$

NUMBER OF ELEMENTS	LEVEL OF MULTIPLICATION	SIGNAL POWER IN OUTPUT CKT.	NOISE POWER
1	1	$\frac{1}{2} A^2$	$\frac{1}{BT}\left[R_n(0)\right]$
2	1	$\frac{1}{4} A^4$	$\frac{1}{BT}\left[\frac{A^2R_n(0)}{2} + \frac{R_n(0)^2}{2}\right].$
4	2	$\frac{9}{64} A^8$	$\frac{1}{BT}\left[\frac{9}{16}A^6R_n(0) + \frac{30}{32}A^4R_n(0)^2 + \frac{9}{16}A^2R_n(0)^3 + \frac{1}{4}R_n(0)^4\right].$
6	3	$\frac{25}{256} A^{12}$	$\frac{1}{BT}\left[\frac{150}{256}A^{10}R_n(0) + \frac{390}{256}A^8R_n(0)^2 + \frac{420}{256}A^6R_n(0)^3 + \frac{40}{32}A^4R_n(0)^4 + \frac{575}{1024}A^2R_n(0)^5 + \frac{78}{160}R_n(0)^6\right].$

VI. NONLINEAR ANTENNAS AND CHANNEL CAPACITY

A direct comparison of the effect on channel capacity of the linear and nonlinear antennas is complicated by the change in dimensionality of the message space effected by the multiplicative and averaging processes. A comparison is possible, however, between the nonlinear antenna system and a linear antenna with an associated square law circuit. The signal-to-noise ratios in the outputs of these two systems can show the effectiveness of the nonlinear antenna system as related to the familiar linear antenna system.

Consider an antenna array of p elements, spaced so the random noise voltages induced on any two elements are uncorrelated. Then, if s is the voltage induced on each element by a signal source in the main lobe, and n_i is the random noise voltage induced on the i-th element, the voltage from the linear array is

$$V_L = ps + \sum_{i=1}^{p} n_i .$$

After squaring and averaging

$$Y_L = \overline{V_L^2} = p^2\overline{s^2} + pR_n(0).$$

But if these p elements are split into two groups of p_1 and p_2 elements respectively ($p_1+p_2 = p$), and the voltages of these two sub-arrays are multiplied and averaged, the output of this nonlinear array becomes:

$$Y_{NL} = \overline{(p_1 s + \sum_{i=1}^{p_1} n_i)(p_2 s + \sum_{j=1}^{p_2} n_j)} = p_1 p_2 \overline{s^2} .$$

The noise power in the output of each system will depend on the type of signal source. Two such sources will be examined: (1) a random signal source with a gaussian distribution, and (2) a simple sinusoid, $s = A \cos t$. Then in terms of the signal to noise power ratio, $\frac{S}{N}$,

1. s gaussian.

Linear Array $$\left(\frac{S}{N}\right)_{out} = BT\left(\frac{S}{N}\right)^2_{in} \left[\frac{1}{\left(\frac{S}{N}\right)^2_{in} + \frac{2}{p}\left(\frac{S}{N}\right)_{in} + \frac{1}{p^2}}\right]$$

Nonlinear Array $$\left(\frac{S}{N}\right)_{out} = BT\left(\frac{S}{N}\right)^2_{in} \left[\frac{2p_1p_2}{2p_1p_2\left(\frac{S}{N}\right)^2_{in} + p\left(\frac{S}{N}\right) + 1}\right]$$

when $p_1 = p_2 = \frac{p}{2}$

$$\left(\frac{S}{N}\right)_{out} = BT\left(\frac{S}{N}\right)^2_{in} \left[\frac{1}{\left(\frac{S}{N}\right)^2_{in} + \frac{2}{p}\left(\frac{S}{N}\right) + \frac{2}{p^2}}\right]$$

2. $s = A \cos \omega t$.

Linear Array $$\left(\frac{S}{N}\right)_{out} = BT\left(\frac{S}{N}\right)^2_{in} \left[\frac{1}{\frac{1}{p^2} + \frac{2}{p}\left(\frac{S}{N}\right)_{in}}\right]$$

Nonlinear Array $$\left(\frac{S}{N}\right)_{out} = BT\left(\frac{S}{N}\right)^2_{in} \left[\frac{2p_1p_2}{1 + p\left(\frac{S}{N}\right)_{in}}\right]$$

when $p_1 = p_2 = \frac{p}{2}$

$$\left(\frac{S}{N}\right)_{out} = BT\left(\frac{S}{N}\right)^2_{in} \left[\frac{1}{\frac{2}{p^2} + \frac{2}{p}\left(\frac{S}{N}\right)_{in}}\right]$$

where $\left(\frac{S}{N}\right)_{in}$ refers to the signal-to-noise power ratio induced on a single element of the array, and $\left(\frac{S}{N}\right)_{out}$ refers to

the signal-to-noise power ratio at the output of the antenna array combining circuit.

When the signal-to-noise ratio and the number of elements are such that their product is large $\left[p(\frac{S}{N})_{in} \gg 1 \right]$, the output signal-to-noise ratio is approximately the same for both arrays:

$$(\frac{S}{N})_{out} \approx BT(\frac{S}{N})_{in} \left(\frac{p}{2}\right).$$

In this case, the channel capacity would be expected to be the same for both arrays if the same number of elements, p, were used in each.

When the signal-to-noise ratio and the number of elements are such that their product is small $\left[p(\frac{S}{N})_{in} \ll 1 \right]$, the output signal-to-noise ratios become:

$$\text{Linear Array} \quad (\frac{S}{N})_{out} \approx BT(\frac{S}{N})^2_{in} \left(p^2\right)$$

$$\text{Nonlinear Array} \quad (\frac{S}{N})_{out} \approx BT(\frac{S}{N})^2_{in} \left(\frac{p^2}{2}\right)$$

Now the channel capacity of a nonlinear array is not so great as that of a linear array with the same number of elements, p.

This analysis has tended to ignore one very important aspect of the nonlinear array. The average output voltage of the linear array (Y_L) will contain a component proportional to the signal and a component proportional to the noise; the average output voltage of the nonlinear array (Y_{NL}) has no component which is independent of the signal. This emphasized the real difference between the linear and nonlinear arrays;

the nonlinear array will generally be capable of a smaller channel capacity than the linear array, but in the final output voltage the effect of the random noise sources can be made arbitrarily small in the nonlinear array, while this is not possible in the linear array.

the signal-to-noise power ratio at the output of the antenna array combining circuit.

When the signal-to-noise ratio and the number of elements are such that their product is large $\left[p(\frac{S}{N})_{in} \gg 1 \right]$, the output signal-to-noise ratio is approximately the same for both arrays:

$$(\frac{S}{N})_{out} \approx BT(\frac{S}{N})_{in}\left(\frac{p}{2}\right).$$

In this case, the channel capacity would be expected to be the same for both arrays if the same number of elements, p, were used in each.

When the signal-to-noise ratio and the number of elements are such that their product is small $\left[p(\frac{S}{N})_{in} \ll 1 \right]$, the output signal-to-noise ratios become:

$$\text{Linear Array} \quad (\frac{S}{N})_{out} \approx BT(\frac{S}{N})^2_{in}\left(p^2\right)$$

$$\text{Nonlinear Array} \quad (\frac{S}{N})_{out} \approx BT(\frac{S}{N})^2_{in}\left(\frac{p^2}{2}\right)$$

Now the channel capacity of a nonlinear array is not so great as that of a linear array with the same number of elements, p.

This analysis has tended to ignore one very important aspect of the nonlinear array. The average output voltage of the linear array (Y_L) will contain a component proportional to the signal and a component proportional to the noise; the average output voltage of the nonlinear array (Y_{NL}) has no component which is independent of the signal. This emphasized the real difference between the linear and nonlinear arrays;

the nonlinear array will generally be capable of a smaller channel capacity than the linear array, but in the final output voltage the effect of the random noise sources can be made arbitrarily small in the nonlinear array, while this is not possible in the linear array.

VII. DESIGN OF NONLINEAR ARRAYS

A. Basic Considerations

The preceeding analyses form the basis for certain conclusions regarding the employment of antenna arrays which utilize nonlinear processes in the combining network:

1. Random noise voltages induced on the antenna elements can be reduced to any desired level by proper element spacing and by averaging for a sufficiently long interval. Noise power is approximately inversely proportional to the averaging interval, BT.

2. Cross multiplication processes in the antenna circuitry will introduce additional noise terms resulting from interaction of signal and noise voltages, of different noise voltages, or of different signal voltages. These added noise terms attain their minimum values when the level of multiplication is limited to one.

3. When the level of multiplication is greater than one, the interaction of two or more independent signal sources will introduce components in the output voltage which can not be reduced by averaging.

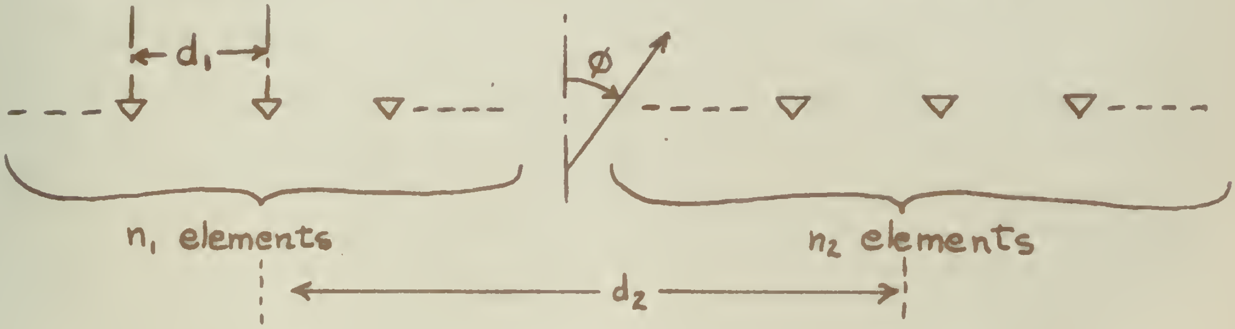
4. The examples of nonlinear antennas considered have demonstrated directional patterns with main lobes significantly narrower than those obtainable from linear arrays of the same size.

5. The nonlinear array is not efficient in a general communication system, but offers particular advantages to a system in which narrow beamwidth is paramount.

These general conclusions will be the basis for the deisgn of nonlinear antenna arrays. The difficulties which arise when the level of multiplication exceeds one outweigh any apparent increase in directivity which might be obtained. Therefore, the combining network will be a combination of addition and multiplication circuits, with the level of multiplication limited to one.

B. Products of Linear Arrays.

If we consider a uniform array of n equally spaced isotropic elements, the simplest type of nonlinear antenna which can be constructed (with level of multiplication equal to one) is that in which the elements are formed into two linear arrays whose output voltages are then multiplied and averaged. There are, of course, a number of different combinations of the elements which can be realized in accomplishing this.



For two groups, containing n_1 and n_2 elements respectively, so that $n = n_1+n_2$, and with spacing d_1 between elements and d_2 between group centers, the directional patterns can be written:

$$P_{n_1}(\Psi) = \frac{\sin n_1(\Psi/2)}{n_1 \sin(\Psi/2)}$$

$$P_{n_2}(\Psi) = \frac{\sin n_2(\Psi/2)}{n_2 \sin(\Psi/2)}$$

and $$P_n(\Psi) = \frac{\sin n_1(\Psi/2)}{n_1 \sin(\Psi/2)} \quad \frac{\sin n_2(\Psi/2)}{n_2 \sin(\Psi/2)} \quad \cos(\frac{d_2}{d_1}\Psi)$$

The two parameters controling $P_n(\Psi)$ are

1. the total number of elements, and
2. the number of elements in one of the groups.

$P_n(\Psi)$ is equal to unity when $\Psi = 0$. Each term in the expression for $P_n(\Psi)$ will introduce zeros into the directional pattern. The first and second terms will haverzeros equally spaced along the Ψ-axis at intervals of $2\pi/n_1$ and $2\pi/n_2$ respectively. There are $\left[(n_1-1)+(n_2-1) = n-2\right]$ such zeros between $\Psi = 0$ and $\Psi = 2\pi$. Assuming there is no overlapping of the two groups, $d_2/d_1 = n/2$; the final term in the directional pattern, $\cos(d_2/d_1)\Psi = \cos n\Psi/2$, has n zeros in the interval between $\Psi = 0$ and $\Psi = 2\pi$.

It is evident then that a uniform nonlinear array with $n_1+n_2 = n$ elements will have a directional pattern with $2(n-1)$ zeros instead of the $(n-1)$ which would be available for a simple n-element linear array.

The $\cos n\Psi/2$ factor in the directional pattern is the important innovation of this type of nonlinear antenna. This factor will cause the pattern to pass through zero for $\Psi = \pi/n$, regardless of the manner of splitting the n elements into two groups. If these n elements were utilized as a uniform linear array, the first zero would occur at $\Psi = 2\pi/n$. Thus the beamwidth of the principal lobe of the nonlinear array is approximately ½ that of a linear array of the same size.

Since the location of n-2 zeros will depend on the choice of n_1 and $n_2 = n - n_1$, a number of directional patterns are possible for any fixed total number of elements, n. Figure 8 compares the directional patterns which result when $n = 6$.

Variation in element spacing could influence all of the terms in the equation for the directional pattern of the array. Figures 9 and 10 demonstrate this effect, showing the directional patterns which result when the n_1 elements are at spacing d_1, while the n_2 elements are at spacing $2d_1$.

In these figures and in other investigations of the directional patterns, it is apparent that the $\cos(d_2/d_1)\Psi$ term caused by separation of group centers has the greatest effect in narrowing the principal lobe.

C. Data Processing by Correlation Circuits.

Data processing by correlation circuits offers an attractive possibility for improving the presentation of the information available from an antenna array. Such a system could be based on the multiplication of the output voltage

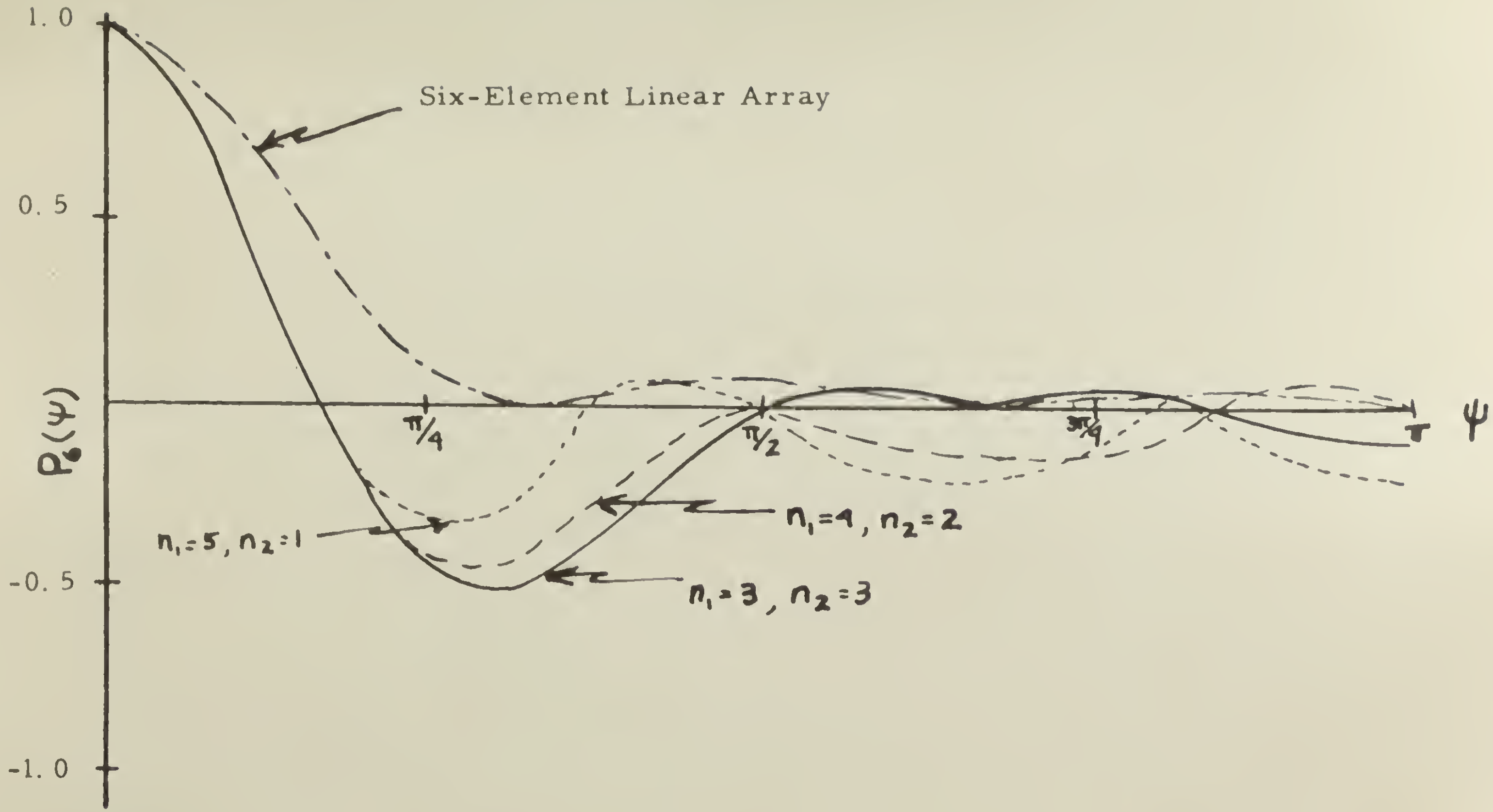


FIG. 8. -- Directivity patterns for a Six-Element nonlinear array.

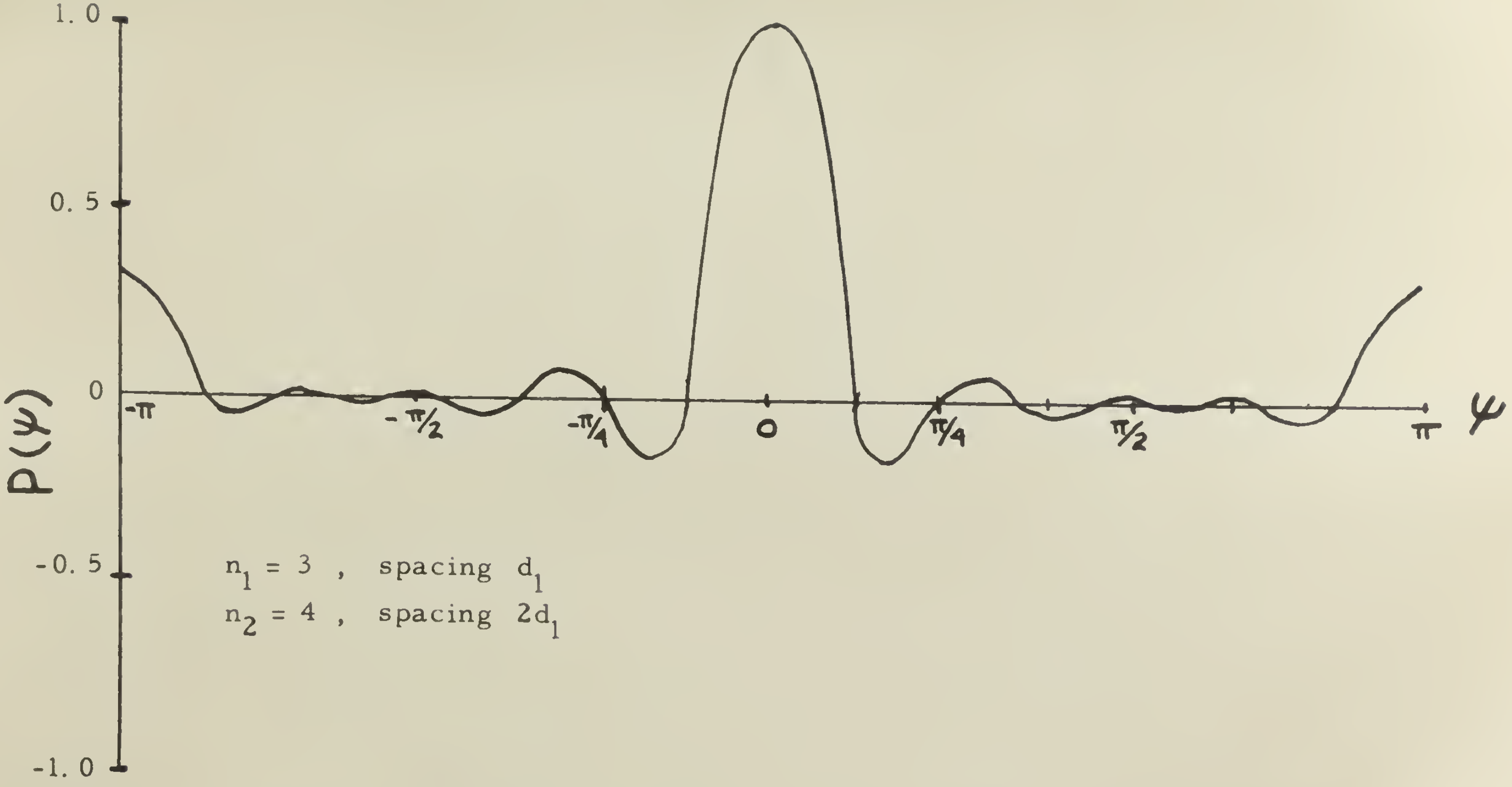


FIG. 9. -- Directivity Pattern for nonlinear array with nonuniform spacing.

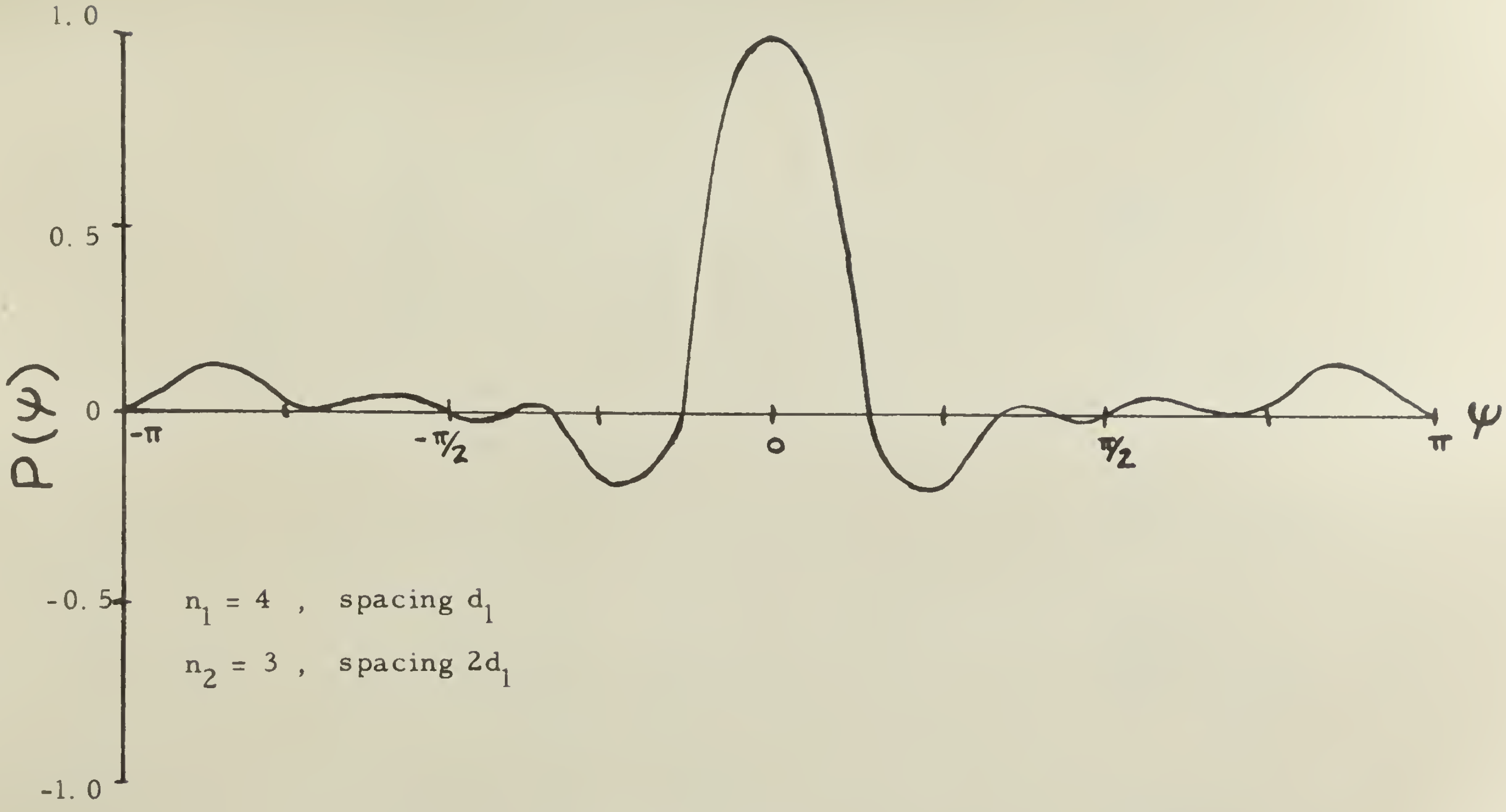
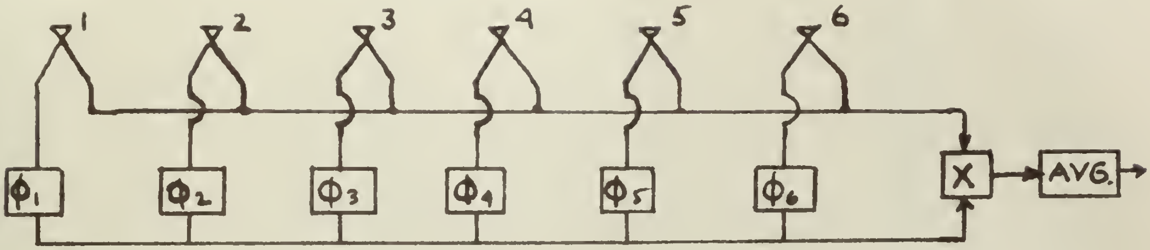


FIG. 10. -- Directivity pattern for nonlinear array with nonuniform spacing.

of one linear array by the output voltage of a second linear array in which a progressive phase shift has been introduced. For example, using again a total of six elements:



By providing a number of phase shifted circuits with the progressive phase shifts chosen to form directional patterns whose maxima occur at the zeros of the unshifted pattern or at the sidelobe maxima of the unshifted pattern, a number of such correlation processes can be carried out. By combining these correlation voltages into a single output voltage, an improved directional pattern is formed.

Figure 11 shows this resulting pattern for the six-element array. The unshifted pattern is given by

$$P_6(\psi) = \frac{\sin 6(\psi/2)}{6 \sin(\psi/2)}$$

The phase shifted patterns are given by

$$P_6^{\pm}(\psi) = \frac{\sin 6 \frac{1}{2}(\psi \pm \alpha)}{6 \sin \frac{1}{2}(\psi \pm \alpha)} .$$

where $\alpha = \pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, $5\pi/6$, and π.

By fully utilizing all of the phase shifted circuits connected to these six elements, it would be possible to provide a number of antenna patterns with main beams displaced (equally in ψ)

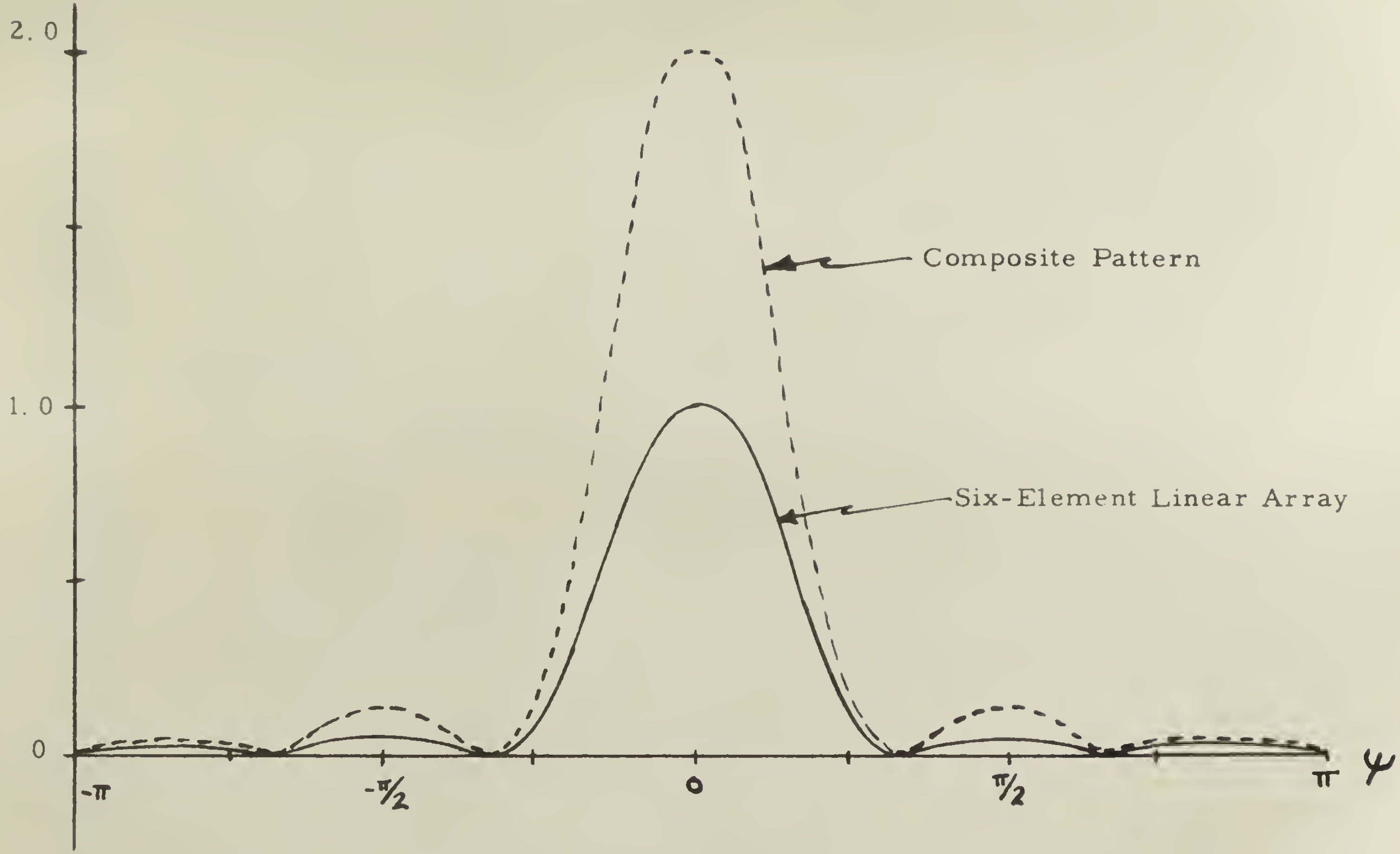
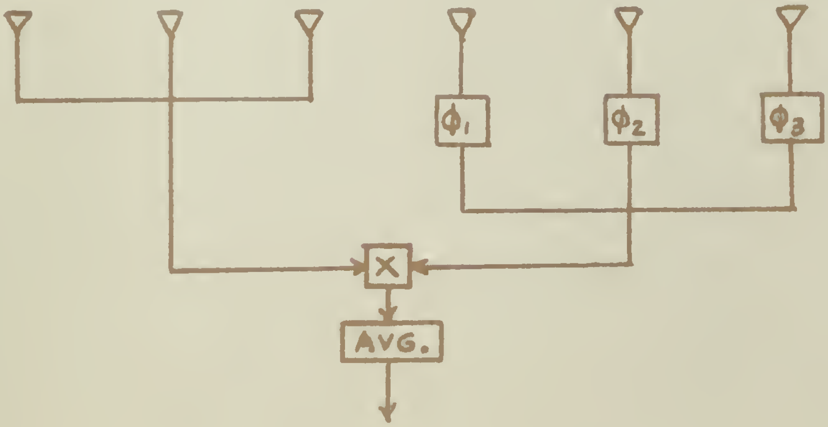


FIG. 11. -- Directivity pattern, six-element array, multiple beam correlation.

throughout 360°, all from a stationary array. This multiple-beam antenna is not new, but the improved beamwidth provided by the correlators permits a refinement in performance without increasing the overall length of the array.

Using the same six elements to provide unshifted as well as phase shifted voltages will not permit elimination of noise voltages by averaging. And it does not take advantage of the beam narrowing feature of the $\cos(d_2/d_1)$ term present when the linear arrays are separated. Thus some improvement in performance would be expected from splitting the elements into two separated linear arrays, with progressive phase shifting applied to one of these resulting arrays. Figure 12 shows the pattern which results when the six elements are formed into two uniformly spaced three-element linear arrays.



$$P_6(\psi) = \frac{\sin 3(\psi/2)}{3\sin(\psi/2)} \, \frac{\sin 3\,\frac{1}{2}(\psi \pm \alpha)}{3\sin\frac{1}{2}(\psi \pm \alpha)} \cos(3\psi \pm \alpha)$$

where $\alpha = 0,\ \pi/3,\ 2\pi/3$, and π.

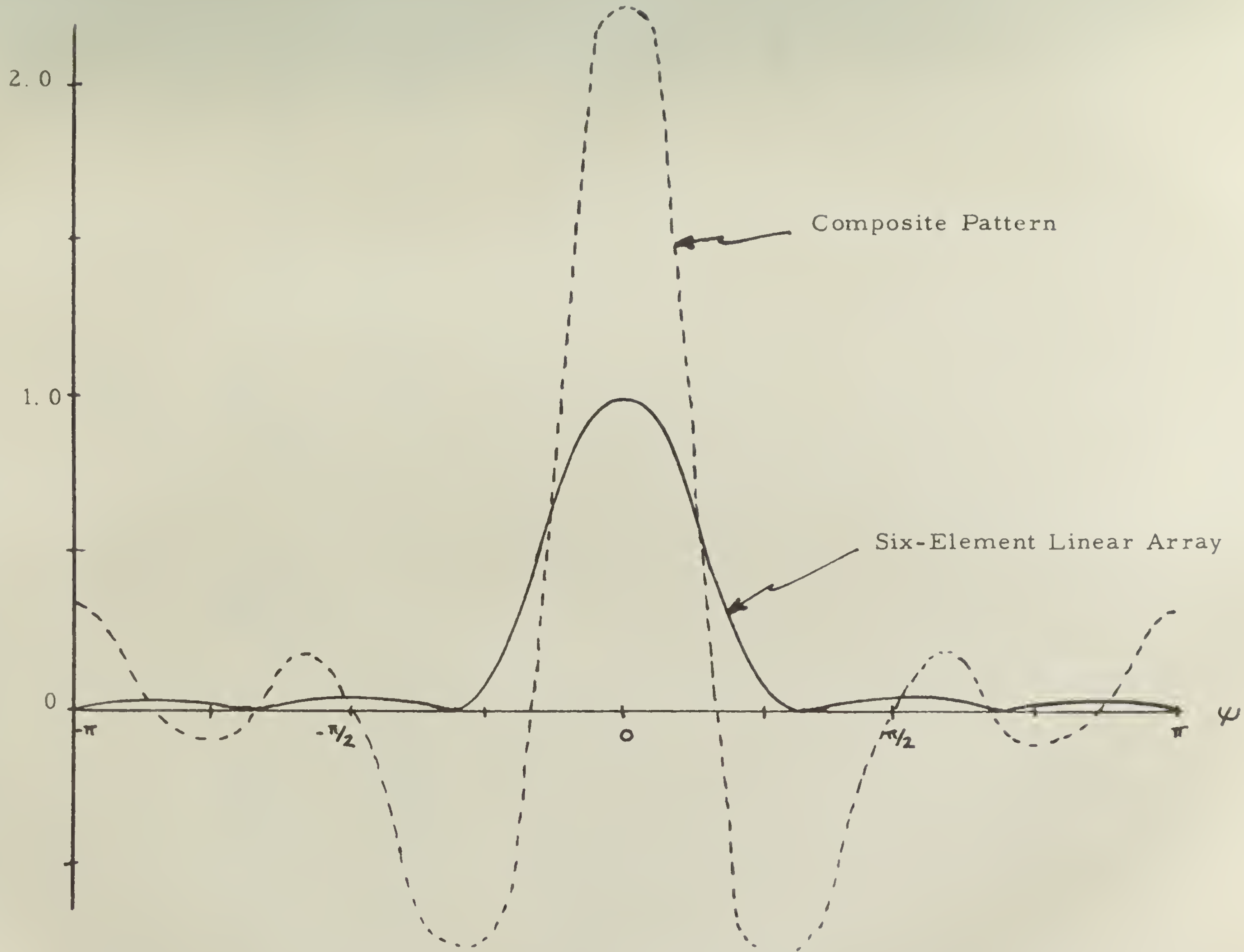


FIG. 12. -- Directivity pattern; two three-element arrays; multiple beam correlation.

APPENDIX I: THE PROBABILITY DISTRIBUTION FOR THE OUTPUT NOISE VOLTAGE OF A TWO-ELEMENT CORRELATION ARRAY.

A random distribution of noise sources surrounding a two-element antenna array will induce voltages which can be represented by stationary time series with gaussian probability distributions. The noise voltages can thus be completely described statistically by specifying their cross- and auto-correlation functions. Each noise source will induce a voltage on one element which is identical to that induced on the other element, but with a time delay factor due to the finite time of travel of the wavefront from one element to the other. As shown previously, the cross- and auto-correlation functions of the total induced voltage can be obtained by summing the contributions of the individual sources. For a uniform distribution of independent noise sources and rectangular band limiting:

$$\Psi_{11}(s) = 4\pi B\eta \frac{\sin \pi Bs}{\pi Bs} \cos \omega_0 s = R_n(0) \frac{\sin \pi Bs}{\pi Bs} \cos \omega_0 s$$

$$\Psi_{12}(s) = R_n(0) \frac{\sin k\frac{\pi Bd}{c}}{k\frac{\pi Bd}{c}} \quad \frac{\sin \frac{\pi Bd}{c}}{\frac{\pi Bd}{c}} \quad \frac{\sin \pi Bs}{\pi Bs} \cos \omega_0 s$$

$$= R_n(\tau) \frac{\sin \pi Bs}{\pi Bs} \cos \omega_0 s$$

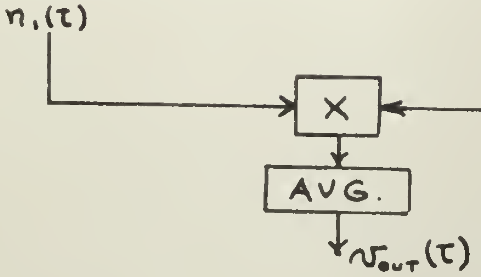
where:

1) noise sources in element of angle $d\Omega$ produce noise power $\eta \, d\Omega$ watts/cps;

2) B is bandwidth, ω_o is center frequency;

3) $k = \frac{\omega_o}{\pi B}$, and d is element spacing.

The elements of the system can be represented as:



When $n_1(t) = n_2(t)$, the circuit is that of a square law detector; the statistics of this system have been studied by several authors[11, 12]. Emerson has shown that the probability distribution of the output voltage of the square law detector is determined by the eigenvalues of a linear, homogeneous, integral equation whose kernel is a function only of the correlation function of the input voltage and the impulse response of the circuit employed as an averager.

Using this general method Lampard[13] has obtained an explicit solution for correlated input voltages, but only for the case of no postmultiplier filtering.

Although it appears that there is no explicit solution to this general antenna noise problem when postmultiplier filtering is employed, the resulting distribution can be represented by an infinite series whose significant terms can be calculated. If these input noise voltages have bandlimited gaussian distributions, the output from the averaging circuit can be expressed in terms of an infinite series and displayed

graphically in a manner similar to that used by Emerson.

The output voltage of the circuit is written as a convolution

$$v_o(t) = \int_0^t n_1(t-u)n_1(t-u)h(u)\, du$$

where h(u) is the impulse response of the averaging circuit, Since, for this antenna situation, $n_1(t)$ differs from $n_2(t)$ only by the time delay factor τ, the output can be written

$$v_o(t) = \int_0^t n_1(t-u)n_1(t+\tau-u)h(u)\, du.$$

The moments of the output distribution can be determined directly:

$$\overline{v_o(t)} = \int_0^t \overline{n_1(t-u)n_1(t+\tau-u)h(u)}\, du.$$

For this perfect averaging circuit

$$h(u) = \begin{cases} \frac{1}{t}, & 0 \leq u \leq t \\ 0, & \text{elsewhere} \end{cases}$$

Then

$$K_1 = \overline{v_o(t)} = \frac{1}{t}\int_0^t \Psi_{12}(0)\, du$$

where $\Psi(s) = \overline{n_i(t)n_j(t+s)}$ are the correlation functions for the bandlimited noise voltages.

$$K_2 = \overline{v_o(t)^2} = \frac{1}{t^2}\int_0^t \int_0^t \overline{n_1(t-u)n_2(t-u)n_1(t-v)n_2(t-v)}\, du\, dv$$

$$= \frac{1}{t^2}\int_0^t \int_0^t \left[\Psi_{12}(0)^2 + \Psi_{11}(u-v)^2 + \Psi_{12}(u-v)^2\right] du\, dv.$$

The variance is $\sigma^2 = K_2 - K_1^2$.

A general expression for the moments could be written in the form of an interated integral

$$K_i = v_o(t)^i = \frac{1}{t^i} \int_0^t \cdot \cdot \int_0^t \overline{n_1(t-x_1)n_2(t-x_1) - - - n_1(t-x_i)n_2(t-x_i)} \, dx_1 - - - dx_i .$$

The output probability distribution can now be determined in terms of a Gram-Charlier series, type A.[14,15] If we introduce the normalized variable

$$y = \frac{v_o(t) - \overline{v_o(t)}}{\sigma} ,$$

then a convenient expansion for the probability distribution of y is given by the series grouped according to Edgeworth:

$$p(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \left\{ 1 + C_3H_3(y) + C_4H_4(y) + C_6H_6(y) + ----- \right\} ,$$

where $H_n(x)$ is the Hermite polynomial of degree n defined by

$$\frac{d^n}{dx^n} (e^{-\frac{1}{2}x^2}) = (-1)^n H_n(x) e^{-\frac{1}{2}x^2} ,$$

where

$$C_i = \frac{\mu_i}{(i!) \quad i}$$

and μ_i is the i-th central moment, obtained from K_i, the i-th moment.

To illustrate the change in the probability distribution of the output voltage with increasing averaging time, let us

consider two isotropic antenna elements in a uniformly distributed noisy medium with an element spacing $d = \frac{\lambda}{4}$. For this element spacing, $R_n(\tau) = \frac{2}{\pi} R_n(0)$.

With no postmultiplier filtering the distribution is given explicitly by Lampard:

$$p(v_o) = \frac{1}{\pi}\left[\Psi_{11}(0)^2 - \Psi_{12}(0)^2\right]^{-\frac{1}{2}} \exp\left\{v_o \frac{\Psi_{12}(0)}{\Psi_{11}(0)^2 - \Psi_{12}(0)^2}\right\} K_o\left\{\frac{|v_o| \; \Psi_{11}(0)}{\Psi_{11}(0)^2 - \Psi_{12}(0)^2}\right\}.$$

where $K_o(x)$ is the modified Bessel function of the second kind and zero order.

With postmultiplier averaging included, the equations for the first three central moments of the distribution are:

$$\mu_1 = \overline{v_o(t)} = \frac{1}{t}\int_0^t R_n(\tau)\, du = R_n(\tau) = \frac{2}{\pi} R_n(0).$$

$$\mu_2 = \sigma^2 = \frac{1}{t^2}\int_0^t\int_0^t \left\{\left[R_n(0)^2 + R_n(\tau)^2\right] \frac{\sin^2 \pi B(u-v)}{\pi B(u-v)^{2}} \cos^2 \omega_o(u-v)\right\} du\, dv.$$

$$\mu_3 = \frac{1}{t^3}\int_0^t\int_0^t\int_0^t \left\{\left[6R_n(0)^2 R_n(\tau) + 2R_n(\tau)^3\right] \frac{\sin \pi B(u-v)}{\pi B(u-v)} \frac{\sin \pi B(v-w)}{\pi B(v-w)} \frac{\sin \pi B(w-u)}{\pi B(w-u)} \cos \omega_o(u-v) \cos \omega_o(v-w) \cos \omega_o(w-u)\right\} du\, dv\, dw.$$

The output probability density functions are now expressed in terms of the Gram-Charlier series. They have been plotted in Figure 13 for selected values of the bandwidth-averaging time

product, Bt. The coefficients have been obtained by neglecting the terms in the $\cos\omega_0(u-v)$ products which represent the high frequency residue. Since in a typical circuit $\omega_0 \gg B$, they are vanishingly small for values of Bt considered here. Actual solution of the integrals to obtain the series coefficients was carried out on a Bendix G-15 digital computer.

For the case Bt = 0 (no averaging) the curve is exact, and the effect of all frequencies is considered.

For Bt = 1, Bt = 2, and Bt = 4, the convergence is fairly rapid (three terms of the series required) for graphical accuracy in regions near the mean. Accuracy for the extremes is not claimed, but would require additional terms in the series.

It will be noted that the tendency toward normality with increasing Bt (increasing average time) is pronounced. This is in general agreement with the results observed by Emerson for the square law detector and would be expected from the general behavior of integrated noise derivable from the Central Limit Theorem.

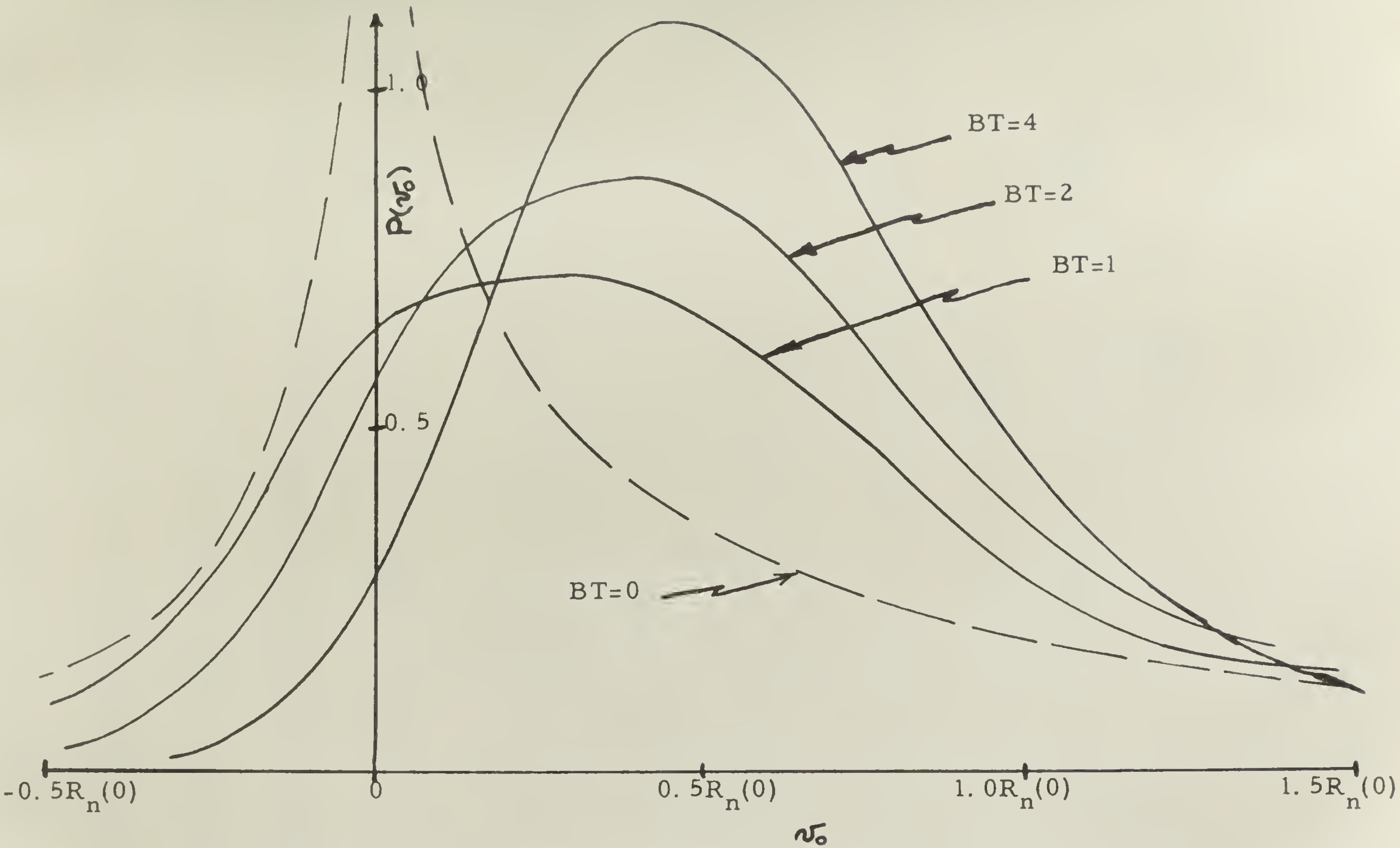
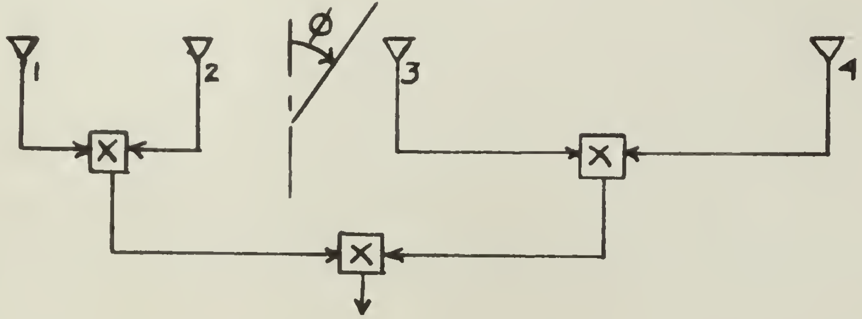


FIG. 13. -- Probability density functions for the output noise voltage of a two-element correlation array for various averaging times. Element spacing $d = \lambda/4$.

APPENDIX II: CALCULATION OF VOLTAGE RESPONSE PATTERNS WITH TWO SIGNAL SOURCES.

The antenna will be a four-element correlation array with relative spacings $d_{12} = D$, $d_{23} = D$, $d_{34} = 2D$, and $D = \lambda/2$. The voltage induced on the i-th element by signal source A is $V_A = A \cos \omega_A (t + \tau_{Ai})$ and that by signal source B is $V_B = B \cos \omega_B (t + \tau_{Bi})$, where $\tau_{Ai} = (d_i/c)\sin \phi_A$ and $\tau_{Bi} = (d_i/c)\sin \phi_B$. The signals which are interesting in this study of correlation processes are grouped into two cases: (1) coherent signals ($\omega_A = \omega_B$) and (2) signals with slightly different frequencies ($\omega_B - \omega_A = \delta\omega$). Therefore, $(\omega_A d_i/c) \approx (\omega_B d_i/c)$ and the following notation can be used:



$$X = \frac{\omega D}{c} \sin \phi_A \qquad \text{and} \qquad Y = \frac{\omega D}{c} \sin \phi_B \;.$$

Then the product of the four-element voltages can be written:

$$V = \Big\{ [A \cos \omega_A t + B \cos \omega_B t][A \cos (\omega_A t + X) + B \cos (\omega_B t + Y)] .$$
$$[A \cos(\omega_A t + 2X) + B \cos (\omega_B t + 2Y)][A \cos (\omega_A t + 4X) + B \cos (\omega_B t + 4Y)] \Big\}$$

Carrying out the indicated multiplications, and rejecting the

multiple frequency components (2ω and 4ω), gives

$$
\begin{aligned}
V = {} & \frac{A^4}{8}[\cos X + \cos 3X + \cos 5X] + \frac{B^4}{8}[\cos Y + \cos 3Y + \cos 5Y] \\
& + \frac{A^3B}{8}[\cos(x+\delta\omega t)+\cos(3X-\delta\omega t)+\cos(5X+\delta\omega t)+\cos(X+4Y+\delta\omega t) \\
& \quad + \cos(X-4Y-\delta\omega t)+\cos(2X+Y+\delta\omega t)+\cos(2X-Y-\delta\omega t)+\cos(3X+2Y+\delta\omega t) \\
& \quad + \cos(3X-2Y-\delta\omega t)+\cos(3X-4Y-\delta\omega t)+\cos(5X-2Y-\delta\omega t)+\cos(6X-Y-\delta\omega t)] \\
& + \frac{AB^3}{8}[\cos(Y-\delta\omega t)+\cos(3Y+\delta\omega t)+\cos(5Y+\delta\omega t)+\cos(Y+4X-\delta\omega t) \\
& \quad + \cos(Y-4X+\delta\omega t)+\cos(2Y+X-\delta\omega t)+\cos(2Y-X+\delta\omega t)+\cos(3Y+2X-\delta\omega t) \\
& \quad + \cos(3Y-2X+\delta\omega t)+\cos(3Y-4X+\delta\omega t)+\cos(5Y-2X+\delta\omega t)+\cos(6Y-X+\delta\omega t)] \\
& + \frac{A^2B^2}{8}[\cos(X+2Y)+\cos(X-2Y)+\cos(X+4Y)+\cos(X-4Y)+\cos(2X+Y)+\cos(2X-Y) \\
& \quad + \cos(2X+3Y)+\cos(2X-3Y)+\cos(3X+2Y)+\cos(3X-2Y)+\cos(4X+Y) \\
& \quad + \cos(4X-Y)+\cos(X-6Y-2\delta\omega t)+\cos(2X-5Y-2\delta\omega t)+\cos(3X-4Y-2\delta\omega t) \\
& \quad + \cos(4X-3Y-2\delta\omega t)+\cos(5X-2Y-2\delta\omega t)+\cos(6X-Y-2\delta\omega t)]
\end{aligned}
$$

Although this is a formidable expression, the capability of the array to resolve the two sources can be estimated by examining two simplified equations:

Case 1. . .when source A is located on the principal lobe of the array ($X = 0$),

Case 2. . .when the principal lobe of the array is directed between the two sources ($X = -Y$).

If it is assumed that $A = B$ and that $\delta\omega = 0$, then the equations for these two cases are:

Case 1

$$
\begin{aligned}
V = \frac{A^4}{8}[6 + 12 \cos Z' + 10 \cos 2Z' + 8 \cos 3Z' + 6 \cos 4Z' \\
+ 4 \cos 5Z' + 2 \cos 6Z']
\end{aligned}
$$

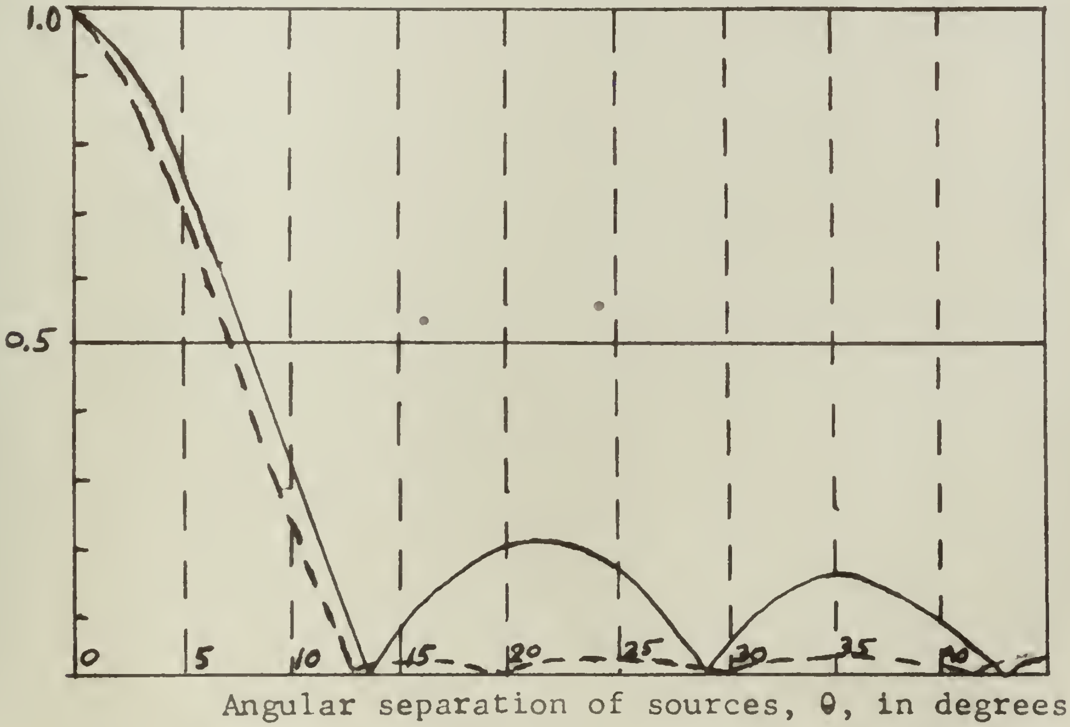
Case 2

$$V = \frac{A^4}{8}[12 \cos Z'' + 12 \cos 3Z'' + 12 \cos 5Z'' + 12 \cos 7Z'']$$

where, with the angular displacement of the sources denoted by θ,

$$Z' = \pi \sin \theta, \qquad \text{and} \qquad Z'' = \pi \sin\frac{\theta}{2}$$

Resolution of the two sources is certainly possible when the magnitude of the voltage response with the main lobe directed between the sources (case 2) is less than the magnitude of the voltage response with the main lobe directed at one of the sources (case 1). Although this is not the only condition under which resolution will occur, a comparison of these two voltage responses as functions of the separation of the sources should indicate the general action of the array in the presence of two sources. Resolution is certain for separations of approximately



VOLTAGE RESPONSE OF ARRAY AS A FUNCTION OF SEPARATION OF SOURCES
Dashed line: main lobe directed at one source
Solid line: main lobe directed between sources

14°, 29°, and 43° where the voltage response for the main lobe directed between the sources drops to zero. It is interesting to note that the voltage response when the main love is directed at one of the sources is uniformly low, and the array will not indicate a maximum voltage response for this orientation. The general voltage response patterns of this array for several specific source separations are shown in the accompanying diagrams, (Figs. 14, 15, and 16).

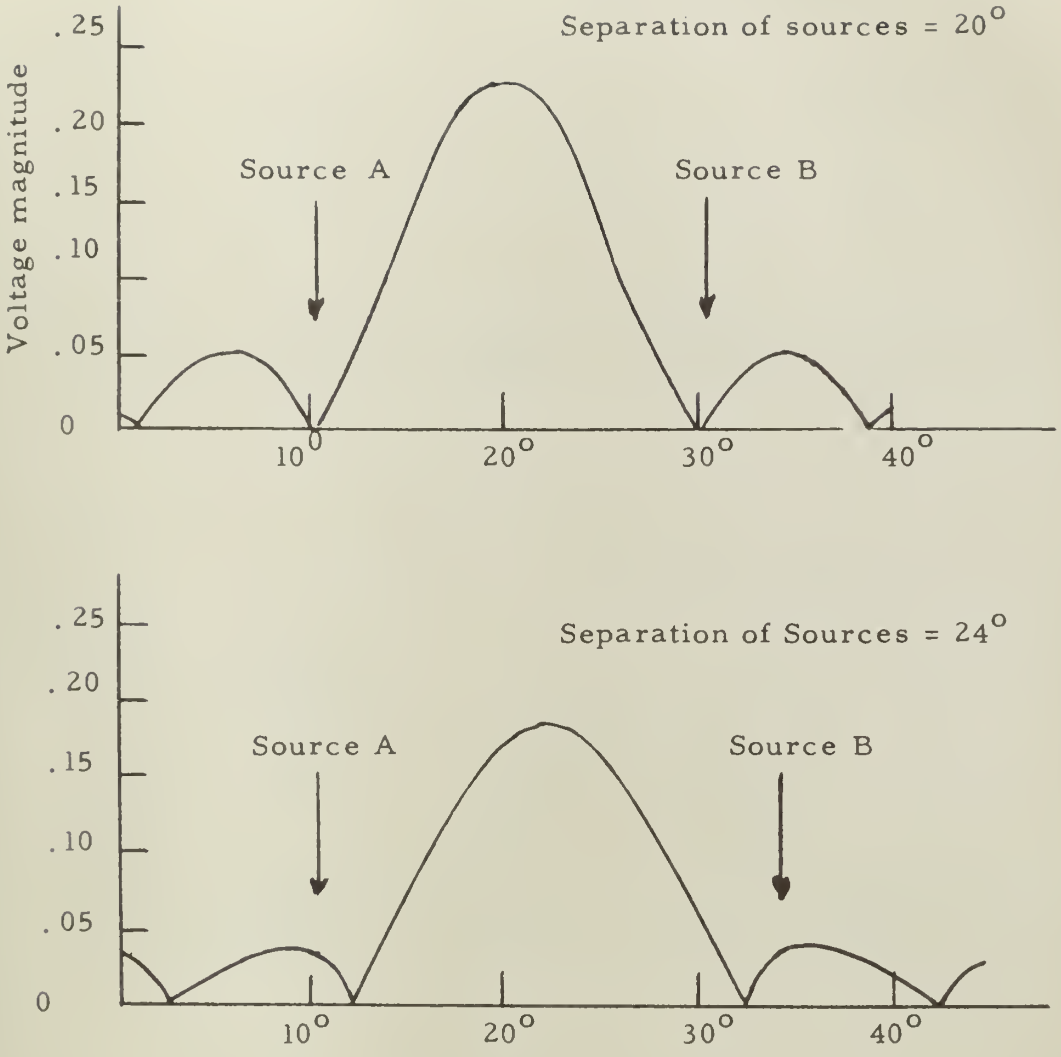


FIG. 14. -- Voltage response pattern, four-element correlation array and two coherent signal sources. Voltage normalized to unity for maximum response of two coincident signal sources.

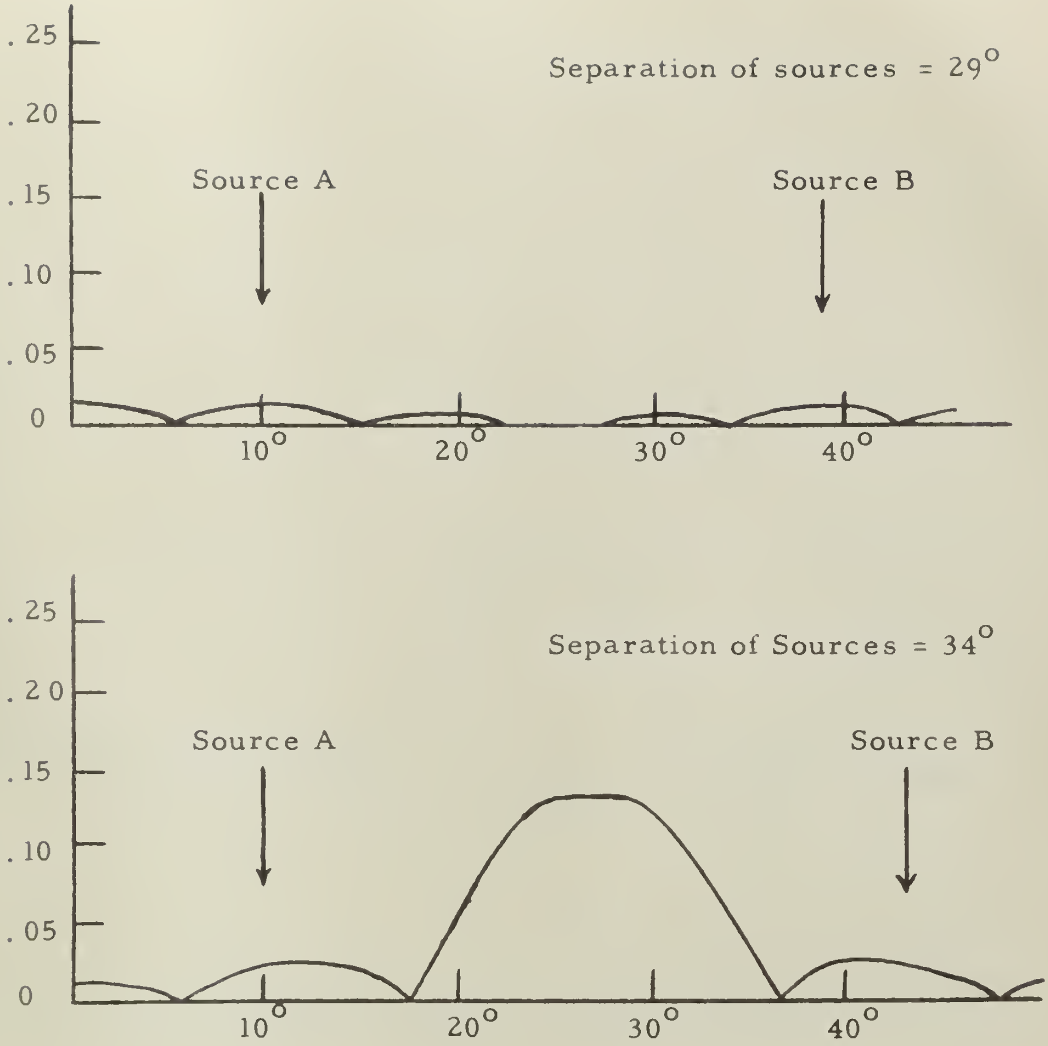


FIG. 15 -- Voltage response patter, four-element correlation array and two coherent signal sources. Voltage normalized to unity for maximum response of two coincident signal sources.

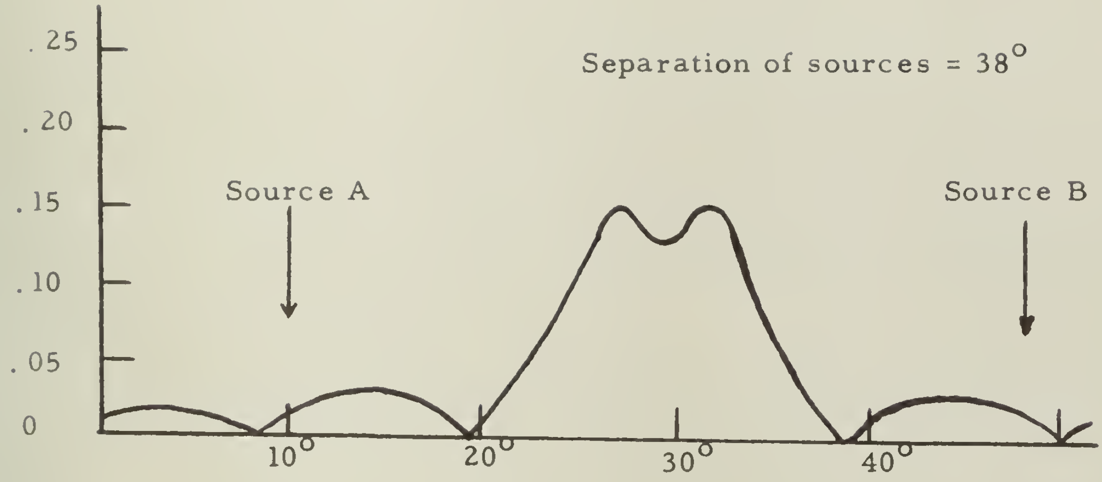


FIG. 16 -- Voltage response pattern, four-element correlation array and two coherent signal sources. Voltage normalized to unity for maximum response of two coincident signal sources.

A similar response pattern, resolving the two sources, can be obtained with a linear antenna with aperture 3λ (6D). This is approximately the same length as the correlation array whose patterns are plotted here.

APPENDIX III: APPLICATION OF CORRELATION TECHNIQUES IN LINEAR ARRAYS.

A. INTRODUCTION

The consideration of the statistical nature of noise and the application of correlation techniques can be utilized in the study of antenna systems which are designed to receive a signal embedded in a general noisy environment. The techniques of correlation theory can be employed to indicate the method of operation of the antenna system which will optimize, in some specified sense, the reception of the desired signal. It is the purpose of this report to summarize some previous results in this field and to indicate a general methodology for attacking the problem.

The general case considered is that of a single signal source, a point source, in a noise medium. No specific restrictions are imposed on the type of signal considered; only a statement of total signal power is made. Thus, the treatment is generally applicable to a signal which is a random function having the same spectrum as the background noise. A signal which is correlated, or whose spectrum varies from that of the background noise would obviously permit further modifications of the receiving techniques which could give improved antenna system performance.

The receiving apparatus considered is a linear broadside array, oriented to put the signal source at the point of maximum sensitivity. Under these conditions the signal-to-noise

ratio of the combined output of the array can be maximized with respect to both of these variables.

The application of correlation techniques to receiving systems has been considered in many forms in the literature. The present method of analysis is based on that introduced by Faran and Hills[1] in their studies of acoustic systems.

B. GENERAL DEVELOPMENT OF THE PROBLEM.

Each element of an array of antennas in a noise field can be considered as an individual source whose output can be suitably modified in amplitude, phase, or time delay before being combined with other source outputs to produce the desired total effect. The output of each such element will be a desired signal voltage plus an undesired noise voltage. If there is a correlation between the noise voltages occurring on any two of the antennas, then it should be possible to modify and combine these voltages in such a manner as to reduce the total effect of the noise in the output of the system.

The correlation coefficient used will be the time average function defined by:

$$R_{12}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e_1(t)e_2(t+\tau)dt$$

When $e_1(t)$ and $e_2(t)$ represent the instantaneous voltages on two antennas, $\overline{e_1{}^2}$ and $\overline{e_2{}^2}$ are proportional to the average powers of the two sources. When these two voltages are added,

the average power of the comgination is proportional to:

$$\begin{aligned}\overline{e^2} &= \overline{[e_1(t_1) + e_2(t_2)]^2} \\ &= \overline{e_1(t_1)^2} + \overline{e_2(t_2)^2} + \overline{2e_1(t_1)e_2(t_2)} \\ &= \overline{e_1(t_1)^2} + \overline{e_2(t_2)^2} + 2R_{12}(t_1 - t_2) \\ &= \overline{e_1(t_1)^2} + \overline{e_2(t_2)^2} + 2R_{12}(\tau)\end{aligned} \qquad (2.2)$$

The cross-correlation coefficient, $R_{12}(\tau)$, will, in general, take on both positive and negative values as τ is varied.

This argument can be extended to the case of n sources. In this case the power of the combined output is proportional to:

$$\overline{e^2} = \overline{\sum_{i=1}^{n} e_i(t_i)^2} = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{e_i(t_i)e_j(t_j)} \qquad (2.3)$$

which will contain [as $R_{ij}(\tau)$] the correlation coefficients between each antenna and all others.

Consider an isotropic antenna located at the origin of the spherical coordinate system, with noise sources distributed on the surface of the enclosing sphere. Then the noise arriving from a differential angle $d\alpha$ in the direction α contributes an amount $f(\alpha)d\alpha$ to the average noise power at the output of the antenna. It will be assumed throughout this report that noise signals arriving from different directions are

independent. The spectral density of the noise power is then expressed as a function of position and frequency:

$$P(\theta, \phi, \omega) = I(\theta, \phi) \cdot W(\omega) \tag{2.4}$$

The actual antenna will not be isotropic but will have a characteristic power gain pattern, $G(\theta, \phi)$, and the spectral density of the noise power $[W(\omega)]$ will be modified $[\text{to } W'(\omega)]$ by the characteristics of the antenna and its associated circuitry (amplifier and phase shifting network).

Since the correlation function (autocorrelation function in this case of a single antenna) and the power spectral density are Fourier transform pairs, the process of determining this correlation function is that of the triple integration:

$$R_{11}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} I(\theta,\phi)G(\theta,\phi)W(\omega)e^{j\omega\tau} \sin\theta \, d\theta \, d\phi \, d\omega \tag{2.5}$$

This method of calculating the correlation coefficient can be readily modified to apply to the case of an array of m antennas. If it is assumed that each of the antennas in the array is identical to each of the others, and, further, if it is assumed that no mutual coupling exists between antennas, then it is only necessary to add:

(1) a term to account for the phase difference (or time delay) in the voltages induced by any given noise source because of the distances between antennas, and

(2) constants A_i and A_j which describe the magnitude of

the gains introduced by the amplifier in each antenna circuit.

$$R_{ij}(\tau) = A_i A_j R'_{ij}(\tau) = \frac{A_i A_j}{2\pi} \iiint I \cdot G \cdot W' e^{jD(\theta\ ,\ \phi\ ,\omega)} e^{j\omega\tau} \sin\theta\, d\theta\, d\phi\, d\omega\ . \tag{2.6}$$

Let the noise voltage received by the i-th antenna be denoted by $n_i(t)$. Then the total noise power of the array is proportional to:

$$N^2 = \overline{\sum_{i=1}^{m} A_i n_i\ (t-\tau_i)^2} \tag{2.7}$$

where A_i is the gain introduced at the i-th antenna, and τ_i is the time delay at this antenna.

$$N^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} A_i A_j\ \overline{n_i(t-\tau_i) n_j(t-\tau_j)} = \sum_{i=1}^{m} \sum_{j=1}^{m} A_i A_j R'_{ij}(\tau_j - \tau_i) \tag{2.3}$$

In order to facilitate interpretation of the mathematical solution of the problem, it is convenient to normalize power equations. Since the antenna elements have been assumed to be identical, $R'_{ij}(\tau)$ depends only on the time delay τ, and $R'(0)$ is the same for all elements. Since

$$R'(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} n^2(t)dt = \overline{n(t)}^2$$

this is a measure of the power induced on a single antenna.

Then, for the system:

$$N_n^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} A_i A_j \frac{R'_{ij}(\tau_j - \tau_i)}{R'(0)} \qquad (2.9)$$

For a linear broadside array with a single signal source located at the point of maximum sensitivity of the array, there is no time delay in the reception of the signal voltages by all elements, so signal power is proportional to:

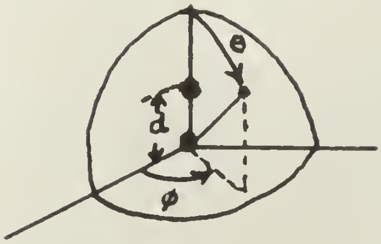
$$S_n^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} A_i A_j = (A_1 + A_2 + \ldots + A_m)^2 \qquad (2.10)$$

If, furthermore, $A_1 + A_2 + \ldots + A_m = 1$, then Eq. (2.9) for N_n^2 is the inverse of the signal-to-noise ratio of the combined output of the array.

C. OPTIMUM PERFORMANCE

For the purpose of this report, optimum performance will be that which gives a maximum signal-to-noise power ratio in the output of the system. Variables to be considered are: (1) the relative gains of the amplifiers connected to each antenna (A_i), (2) a time delay in each antenna circuit (τ_i), and (3) antenna spacing (d).

First, it is necessary to examine the nature of the correlation coefficient, $R_{ij}(\tau)$. Consider a linear array of m identical, evenly-spaced isotropic antennas.



Assume a continuous, uniform distribution of white noise sources over the surface of an enclosing sphere.

Then

$G(\theta, \phi) = 1/4\pi$

$I(\theta, \phi) = 1/4\pi$

$W(\omega) = \frac{\eta}{2}$ watts/cps

Let each amplifier be band limited with identical rectangular pass bands, $\omega_0 + \Delta\omega$.

$$R'_{ij}(\tau) = \frac{\eta}{64\pi^3}\left\{\int_{-\omega_0-\Delta\omega}^{-\omega_0}\int_0^{2\pi}\int_0^{\pi} \sin\theta \, e^{-j\frac{\omega d}{c}\cos\theta} e^{j\omega\tau} \, d\theta d\phi d\omega \right.$$

$$\left. + \int_{\omega_0}^{\omega_0+\Delta\omega}\int_0^{2\pi}\int_0^{\pi} \sin\theta \, e^{-j\frac{\omega d}{c}\cos\theta} e^{j\omega\tau} \, d\theta \, d\phi \, d\omega \right\} \qquad (3.1)$$

$$R_{ij}(\tau) = \frac{\eta}{16\pi^2}\left\{\int_{-\omega_0-\Delta\omega}^{-\omega_0} \frac{\sin \omega d/c}{\omega d/c} + e^{j\omega\tau} \, d\omega + \int_{\omega_0}^{\omega_0+\Delta\omega} \frac{\sin \omega d/c}{\omega d/c} e^{j\omega\tau} d\omega\right\} (3.2)$$

This is solved to give:

$$\frac{R'(d,\tau)}{R'(0)} = \frac{\lambda(\omega)}{4\pi d\cdot\Delta\omega}\left\{S_i\left[\left(\frac{\omega_0+\Delta\omega}{\omega_0}\right)\left(\frac{2\pi d}{\lambda}+\omega_0\tau\right)\right] - S_i\left[\frac{2\pi d}{\lambda}+\omega_0\tau\right]\right.$$

$$\left.+\, S_i\left[\left(\frac{\omega_0+\Delta\omega}{\omega_0}\right)\left(\frac{2\pi d}{\lambda}-\omega_0\tau\right)\right] - S_i\left[\frac{2\pi d}{\lambda}-\omega_0\tau\right]\right\} \qquad (3.3)$$

Under the condition $\Delta\omega \ll \omega_0$, this simplifies to

$$\frac{R'(d,\tau)}{R'(0)} \approx \frac{\sin 2\pi d/\lambda}{2\pi d/\lambda}\cos\omega_0\tau \qquad (3.4)$$

Thus, the correlation coefficient can take on both positive and negative values; then, if the magnitude of the correlation coefficient is established by the element spacing, a time delay network can be utilized to obtain an effective correlation coefficient of any lesser absolute value. In this case, of course, the effect of this time delay on the desired signal voltage must be considered in evaluating the system performance.

Maximizing the output signal-to-noise ratio with respect to the gain introduced at each antenna amplifier or with respect to element spacing is a matter of applying the Lagrange multiplier method to Eq. (2.9) subject to appropriate constraints on total amplification or total array length. The desired quantities are found by solving a set of m+1 simultaneous linear equations in m+1 unknowns (for an array of m elements). Symmetry can be used to reduce the number of equations and unknowns considered.

D. EXAMPLE: FIVE-ELEMENT ARRAY, ISOTROPIC ANTENNAS

Assumptions:

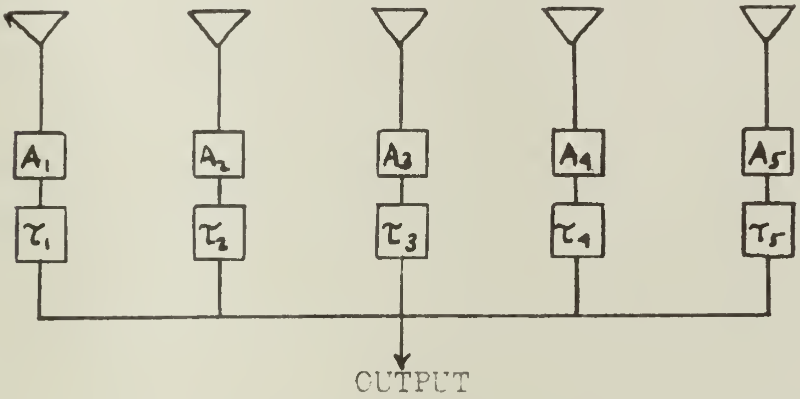
1) Five identical isotropic antennas in a linear broadside array.

2) Single point signal source located in the direction of the maximumnof the array radiation pattern.

3) White noise sources uniformly distributed over the surface of an enclosing sphere.

4) No mutual coupling between antennas.

5) Antenna amplifiers have identical rectangular band pass characteristics $\omega_0 + \Delta\omega$, and $\Delta\omega = 0.01\,\omega_0$.



Case I.--Uniform spacing, (d), between antennas.

$$\begin{aligned} N^2 = \Big\{ & A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + 2R_{12}(d_{12})A_1A_2 \\ & + 2A_1A_3R_{13}(d_{13}) + 2A_1A_4R_{14}(d_{14}) + 2A_1A_5R_{15}(d_{15}) \\ & + 2A_2A_3R_{23}(d_{23}) + 2A_2A_4R_{24}(d_{24}) + 2A_2A_5R_{25}(d_{25}) \\ & + 2A_3A_4R_{34}(d_{34}) + 2A_3A_5R_{35}(d_{35}) + 2A_4A_5R_{45}(d_{45}) \Big\} \end{aligned} \tag{4.1}$$

and

$$S_n^2 = (A_1 + A_2 + A_3 + A_4 + A_5)^2 = 1$$

For a symmetrical array ($A_2 = A_4$, $A_1 = A_5$), and with $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$:

$$N_n^2 = A_3^2 = 2A_2^2 + 2A_1^2 + 4A_1A_2R_{12}(d) + 4A_2A_3R_{23}(d) \quad (4.3)$$

$$+ 4A_1A_3R_{13}(2d) + 4A_1A_2R_{12}(3d) + 2A_2^2R_{22}(2d)$$

$$+ 2A_1^2R_{11}(4d)$$

Lagrange multipliers give four equations in four unknowns:

$$\frac{\delta N_n^2}{\delta A_1} + \frac{\delta S_n^2}{\delta A_1} = 0$$

$$\frac{\delta N_n^2}{\delta A_2} + \frac{\delta S_n^2}{\delta A_2} = 0$$

$$\frac{\delta N_n^2}{\delta A_3} + \frac{\delta S_n^2}{\delta A_3} = 0$$

$$S_n^2 = 1$$

The correlation coefficient as a function of element spacing, d, can be developed as in Section III.

$G(\theta, \phi) = 1/4\pi$

$I(\theta, \phi) = 1/4\pi$

$W(\omega) = \frac{\eta}{2}$ watts/cps

$T = 0$

$$\frac{R'(d,\ 0)}{R'(0,\ 0)} = \frac{\omega\lambda}{2\pi d\cdot\Delta\omega}\left\{S_i\left[\frac{d}{c}\,(\omega_0+\Delta\omega)\right] - S_i\left[\frac{d}{c}\,\omega_0\right]\right\} \tag{4.5}$$

$$= \frac{\lambda(100)}{2\pi d}\left\{S_i\left[1.01\,\frac{\omega_0 d}{c}\right] - S_i\left[\frac{\omega_0 d}{c}\right]\right\}$$

Again, in some cases, this sine-integral expression can be approximated by

$$\frac{\sin \omega_0 d/c}{\omega_0 d/c}$$

The two expressions are equal for at least two significant figures. This is shown in Fig. 17.

For spacing (d) less than a half-wavelength, the amplifications of individual antenna voltages which will result in a maximum signal-to-noise ratio in the combined output differ significantly from the uniform amplification system. For spacings of a half-wavelength or greater, optimum amplification patterns which result from the equaltions above give only a minor improvement in signal-to-noise ratio over that obtained with uniform amplification. Figure 2 shows this signal-to-noise ratio as a function of element spacing for the optimum amplification case and for the uniform amplification case.

Two observations can be made from an examination of Fig. 18:

1) For this case of uniform spacing, maximum signal-to noise ratio occurs for a spacing, d, of 315° between elements. Reference to Fig. 1 shows that for this spacing the cross-correlation coefficients between all antennas (d_1 = 315°, d_2 =630°, d_3 = 945°, d_4 = 1260°) are either negative or negligibly small.

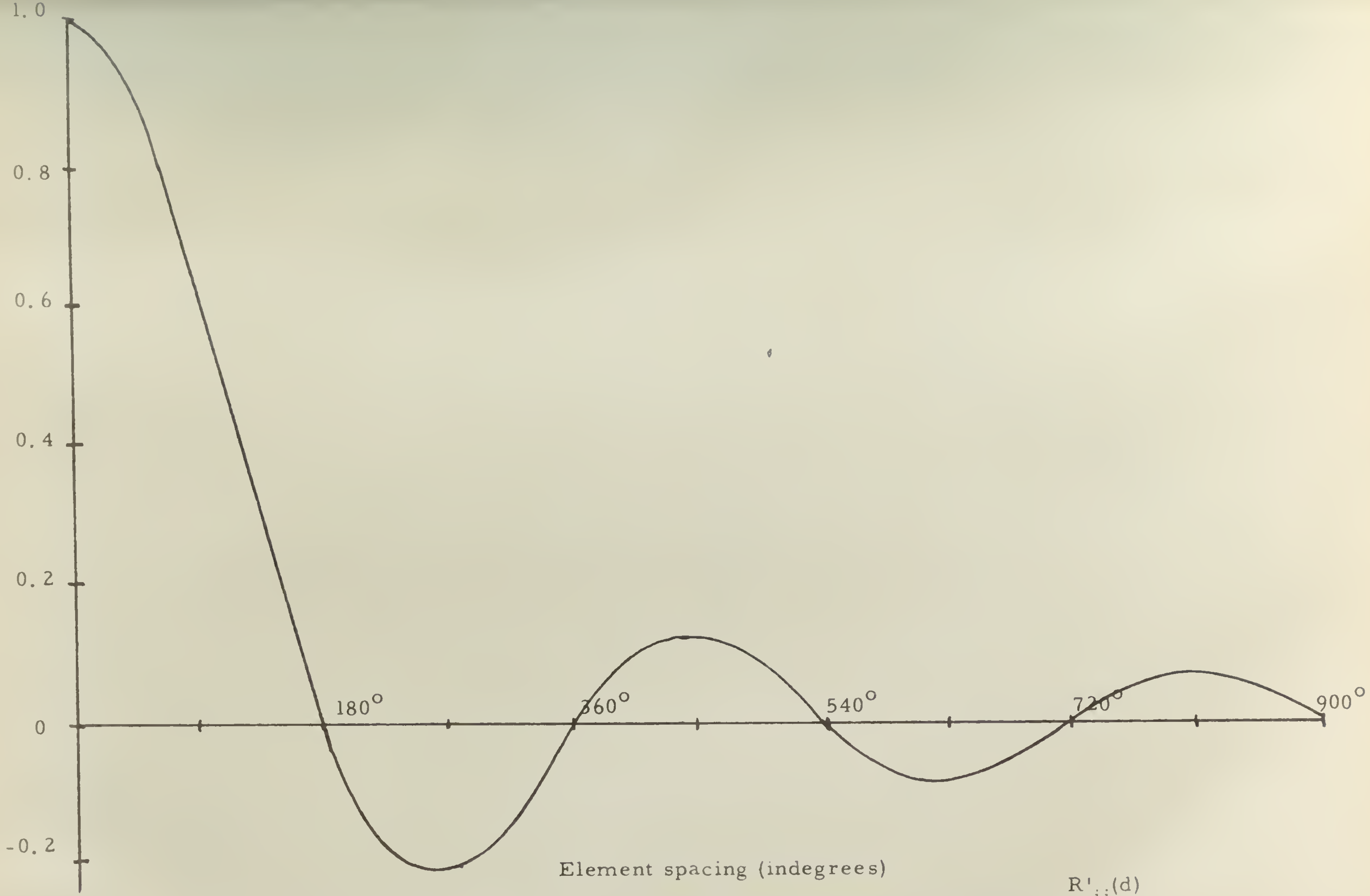


FIG. 17 -- Five-element array, uniform spacing; normalized correlation coefficient $\frac{R'_{ij}(d)}{R'_{ij}(0)}$

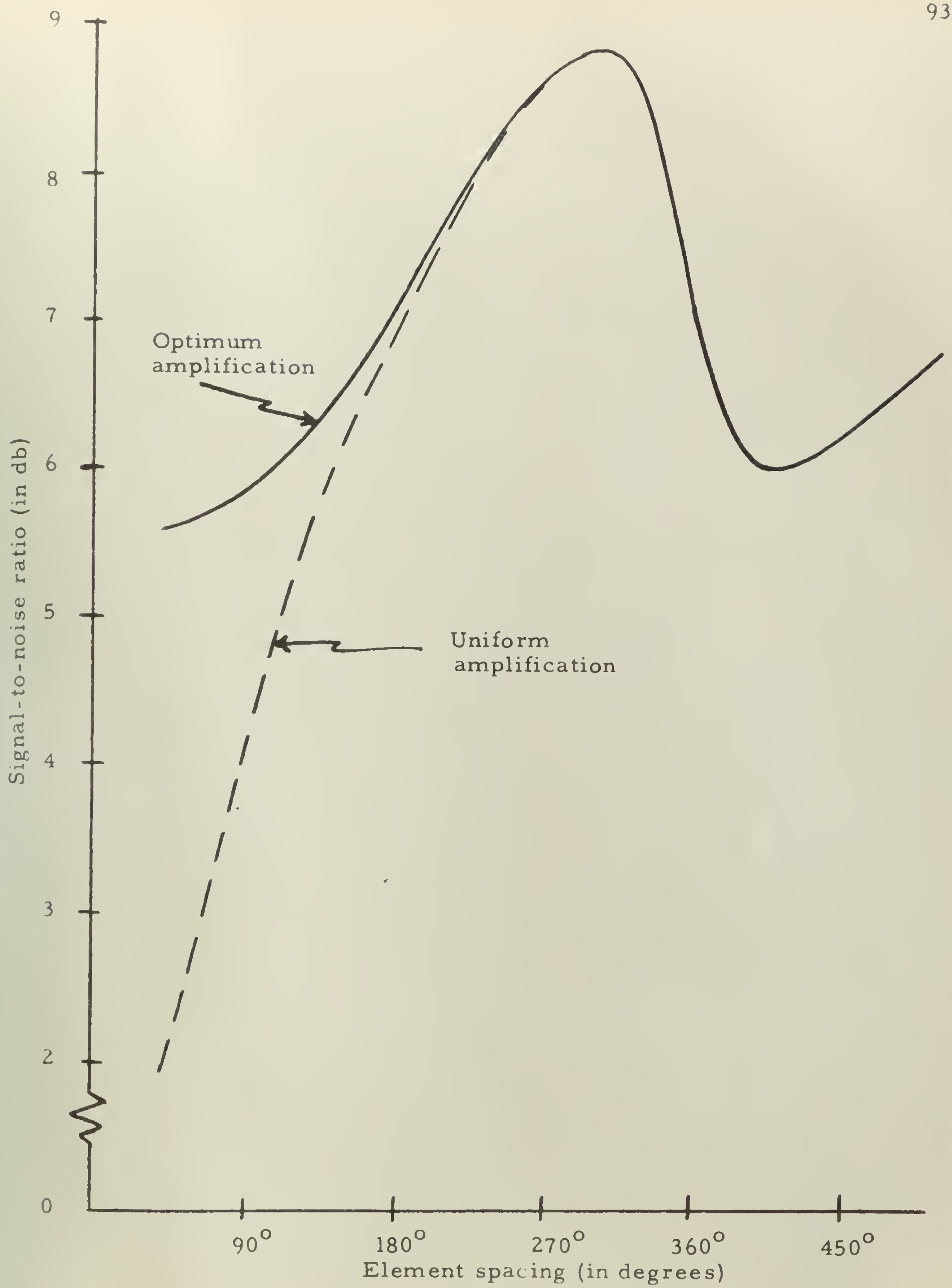


FIG. 18 -- Five-element array, uniform spacing, isotropic antennas, single signal source, spherically distributed noise (0 db corresponds to signal=to-noise ratio of unity on a single element).

Thus, the effect of the cross-components of noise power is to decrease the noise power intercepted by each of the five antennas individually.

2) A significant increase in signal-to-noise ratio can be attained at close spacings by utilizing the amplification patterns resulting from a solution of Eq. (4.4). The rapid increase in individual antenna voltage amplifications for optimum performance as the spacing is narrowed is shown in Fig. 19 and listed in detail in Table 1. Calculations for spacing in this region involve small differences of large quantities so the sine integrals cannot be approximated by the $\frac{\sin x}{x}$ function. As the spacing becomes small, optimum performance depends on precise adjustment of the gains of the individual

Element Spacing	Amplifications				
	A_1	A_2	A_3	A_4	A_5
30°	+ 57.5	-212.9	311.7	-212.9	57.5
45°	13.3	- 44.9	64.2	- 44.9	13.3
60°	4.03	-11.56	16.1	-11.56	4.03
75°	1.81	-4.197	5.77	-4.197	1.81
90°	0.98	- 1.68	2.40	- 1.68	0.98
120°	0.417	-0.214	0.595	-0.214	0.417
150°	0.253	+0.116	0.264	+0.116	0.253
180°	0.199	+0.201	0.199	+0.201	0.199

TABLE 1.--Individual antenna voltage amplification to produce maximum signal-to-noise rationfor the given element spacing.

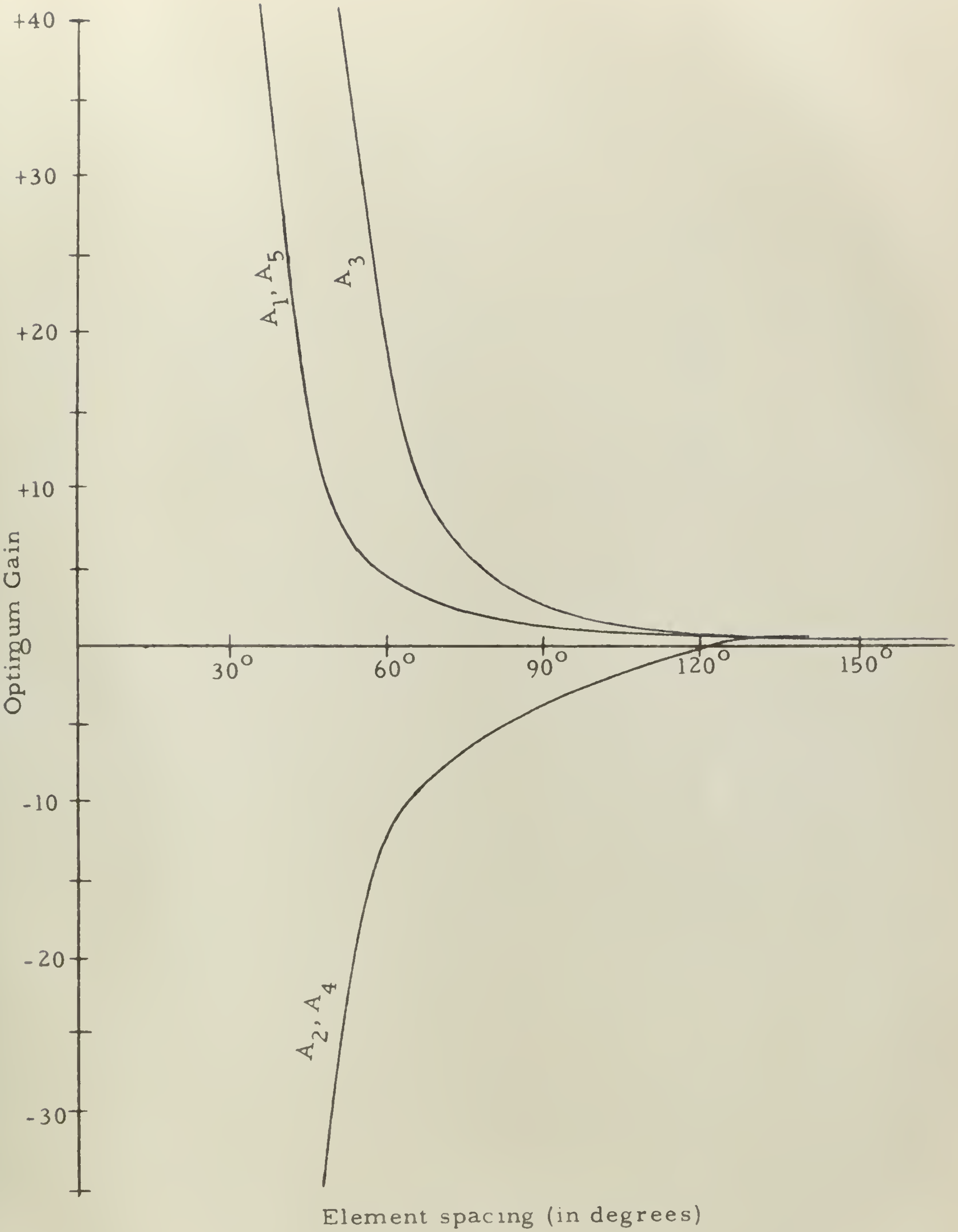
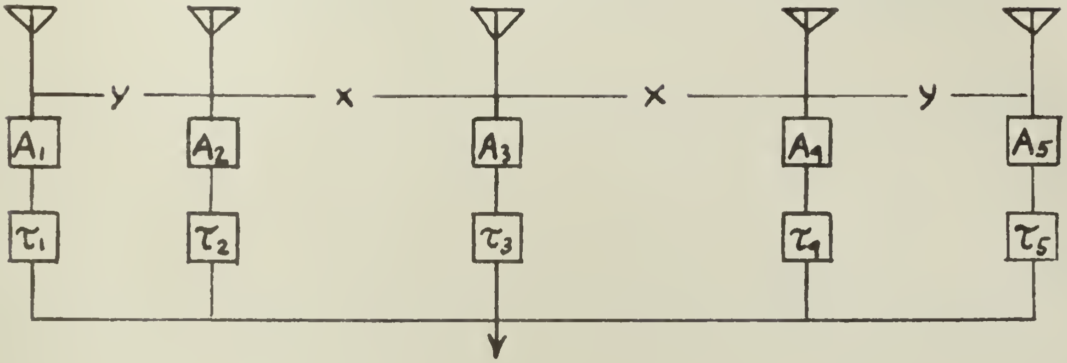


FIG. 19 -- Five-element array, uniform spacing, amplification of antenna voltages to produce maximum signal-to-noise ratio.

antenna amplifiers, and, therefore, becomes increasingly difficult to attain from a practical point of view. This is analogous to the problems of close spacing in super-gain antennas.



Case II.--Variable spacing between antennas.

Consider five isotropic elements in a symmetrical linear broadside array with unequal element spacing. Assuming, as before, that $A_1 = A_5$, $A_2 = A_4$, and $\tau_1=\tau_2=\tau_3=\tau_4=\tau_5=0$, the equations defining noise power and signal power become:

$$N_n^2 = \Big\{A_3^2 + 2A_2^2 + 2A_1^2 + 4A_2A_3R_{23}(x) + 4A_1A_2R_{12}(y) + 4A_1A_2R_{13}(x+y) + 4A_1A_2R_{12}(2x+y) + 2A_1^2R_{11}(2x+2y) + 2A_2^2R_{22}(2x)\Big\} \qquad (4.6)$$

$$S_n^2 = (A_3 + 2A_1 + 2A_2)^2 = 1 \qquad (4.7)$$

Now if it is further assumed that overall length of the array is great enough to permit approximation of the sine integral functions by $\frac{\sin x}{x}$ and to permit use of a uniform gain pattern on the antenna amplifiers, the Lagrange multiplier method can be readily employed to obtain optimum spacings.

$$A_1 = A_2 = A_3 = A_4 = A_5 = 0.20 \qquad (4.8)$$

$$N_n^2 = 0.20 + 0.16\frac{\sin x}{x} + 0.16\frac{\sin y}{y} + 0.16\frac{\sin (x+y)}{(x+y)} \qquad (4.9)$$

$$+ 0.16\frac{\sin(2x+y)}{2x+y} + 0.08\frac{\sin(2x)}{2x} + 0.08\frac{\sin(2x+2y)}{2x+2y}$$

The constraint that must now be imposed is the total length of the array:

$$x + y = K \qquad (4.10)$$

These conditions lead to three equations in three unknowns:

$$\frac{\delta N_n^2}{\delta x} + \lambda \frac{\delta K}{\delta x} = 0$$

$$\frac{\delta N_n^2}{\delta y} + \lambda \frac{\delta K}{\delta y} = 0 \qquad (4.11)$$

$$x + y = K$$

These are readily solved to give the following criteron for extrema of the signal-to-noise ratio as a function of element spacing:

$$\left(\frac{\sin x}{x}\right)' - \left(\frac{\sin y}{y}\right)' + \left(\frac{\sin(2x=y)}{2x+y}\right)' + \left(\frac{\sin 2x}{2x}\right) = 0 \quad (4.12)$$

where the prime indicates the first derivative of the function with respect to the total argument. In general, roots of this equation will give spacings for minima as well as maxima of the signal-to-noise ratio, so it is necessary to carefully select

the correct roots. This equation is, of course, a specific application of a general equation governing a five-element linear array which can be expressed as

$$R'(x) - R'(y) + R'(2x+y) + R'(2x) = 0$$

where R'( ) is the derivative of the normalized correlation coefficient with respect to its total parameter.

The maximum signal-to-noise ratio obtainable by varying element spacing as a function of the total array length is shown in Fig. 20. Spacings which give this maximum ratio are shown for several array lengths in Table 2. Again it is evident that optimum spacings results from cross-correlation coefficients which act to reduce the total noice power of the array.

Array Length	Spacings x	 y	Signal-to-noise ratio (db)
360°	172.0°	188.0°	7.086
450°	215.5°	234.5°	7.891
540°	285.5°	234.5°	8.511
630°	320.0°	310.0°	8.863
720°	282.5°	437.5°	7.536
	492.5°	227.5°	7.320
810°	244.0°	566.0°	7.594
	562.0°	248.0°	7.759
900°	281.5°	618.5°	8.025
	646.5°	253.5°	8.083

TABLE 2.--Element spacings giving maximum signal-to-noise ratio for various total array lengths (note two possible solutions for spacings of 720°, 810°, and 900°).

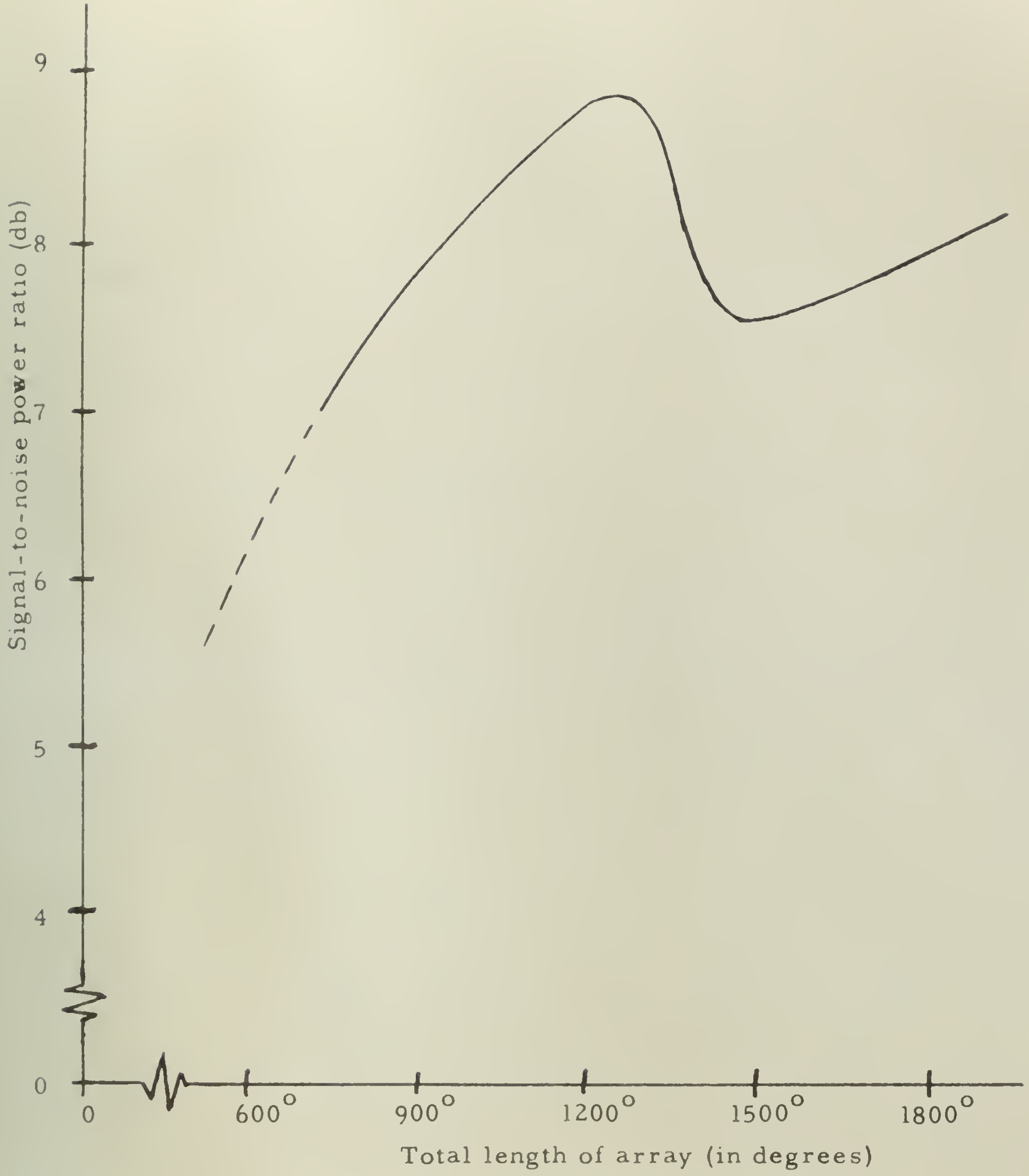


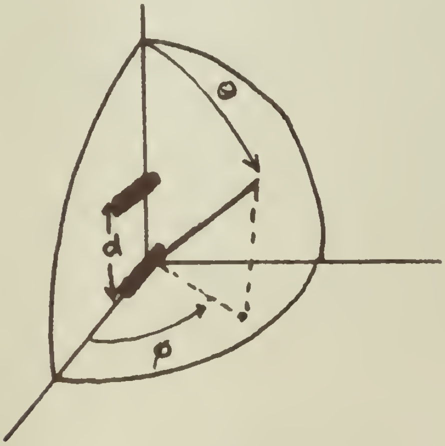
FIG. 20 -- Five-element array, variable spacing for fixed length, spacing adjusted for maximum signal-to-noise ratio.

The curve passes through a definite maximum which corresponds in total array length and in resulting signal-to-noise ratio to the optimum condition discussed in Case I and shown in Fig. 2.

In both cases considered in this example, solution of equations developed in Sections II and III under the conditions of the assumed model have led to definite conditions on the array length, element spacing, or antenna voltage amplification which result in a maximum of the signal-to-noise ratio in the output of the total system.

E. EXAMPLE: FIVE-ELEMENT ARRAY; SHORT DIPOLES

Retain the same assumptions made in the previous example with a change from isotropic antennas to short dipoles. The dipoles can be oriented with their axes perpendicular to the line of the array or parallel to the line of array. Both cases will be considered.



$$I(\theta,\phi) = 1/4\pi$$

$$G(\theta,\phi) = \frac{3}{2}(1 - \sin^2\theta\cos^2\phi)$$

$$W(\omega) = \frac{\eta}{2}$$

Case I.--Axes of dipoles at right angles to the line of the array.

$$R'_{ij}(d,\tau) = \frac{\eta}{16\pi^2}\left\{\int_{-\omega_0-\Delta\omega}^{-\omega_0}\int_0^{2\pi}\int_0^{\pi}\frac{3}{2}(1-\sin^2\theta\cos^2\phi)e^{j\frac{\omega d}{c}\cos\theta}e^{j\omega\tau}\sin\theta\, d\theta\, d\phi\, d\omega\right.$$

$$\left.+\int_{\omega_0}^{\omega_0+\Delta\omega}\int_0^{2\pi}\int_0^{\pi}\frac{3}{2}(1-\sin^2\theta\cos^2\phi)e^{j\frac{\omega d}{c}\cos\theta}e^{j\omega\tau}\sin\theta\, d\theta\, d\phi\, d\omega\right\} \qquad (5.1)$$

Normalizing and integrating gives:

$$\frac{R'_{ij}(d,\tau)}{R'(0)} = \frac{3}{2}\left[\frac{\sin\frac{\omega_0 d}{c}}{\frac{\omega_0 d}{c}} - \frac{1}{3}\Lambda_{3/2}\left(\frac{\omega_0 d}{c}\right)\right](\cos\omega_0\tau) \qquad (5.2)$$

where $\Lambda_{3/2}\left(\frac{\omega_0 d}{c}\right)$ is the Bessel function defined by

$$\Lambda_\nu(x) = \Gamma(\nu+1)\,\frac{J_\nu(x)}{\left(\frac{x}{2}\right)^\nu}$$

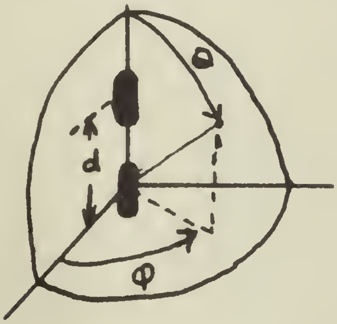
In this integration it has been assumed that the rectangular pass band of the amplifiers is so narrow that

$$\int_{\omega_0}^{\omega_0+\Delta\omega} f(\omega)\, d\omega \approx f(\omega_0)\Delta\omega .$$

The normalized correlation coefficient has been plotted

for T=0 and for variable spacing, d, in Fig. 21.

The signal-to-noise ration (in db) of this array of short dipoles is plotted as a function of element spacing in Fig. 22. As spacing is increased, this signal-to-noise ratio passes through a maximum corresponding again to the spacing which results in a negative correlation coefficient between the noise voltage on any antenna and the noise voltages on all other antennas of the array.



$$I\ (\theta,\ \phi) = 1/4\pi$$

$$G\ (\theta,\ \phi) = \frac{3}{2}\sin^2\theta$$

$$W\ (\omega\ ) = \frac{\eta}{2}$$

Case II.--Axes of dipoles in line with the array.

$$R'_{ij}(d,\tau) = \frac{\eta}{16\pi^2}\left\{\int_{-\omega_0-\Delta\omega}^{\omega_0}\int_0^{2\pi}\int_0^{\pi}\frac{3}{2}\sin^3\theta e^{j\frac{\omega d}{c}\cos\theta}e^{j\omega\tau}\,d\theta d\phi d\omega \right. \tag{5.3}$$

$$\left. + \int_{\omega_0}^{\omega_0+\Delta\omega}\int_0^{2\pi}\int_0^{\pi}\frac{3}{2}\sin^3\theta\ e^{j\frac{\omega d}{c}\cos\theta}e^{j\omega\tau}\,d\theta d\phi d\omega\right\}$$

Normalizing and integrating, again taking advantage of the narrow bandwidth to approximate the integration with respect to ω, gives:

$$\frac{R'_{ij}(d,\tau)}{R'(0)} = \Lambda_{3/2}\left(\frac{\omega_0 d}{c}\right)\cos\omega_0\tau \tag{5.4}$$

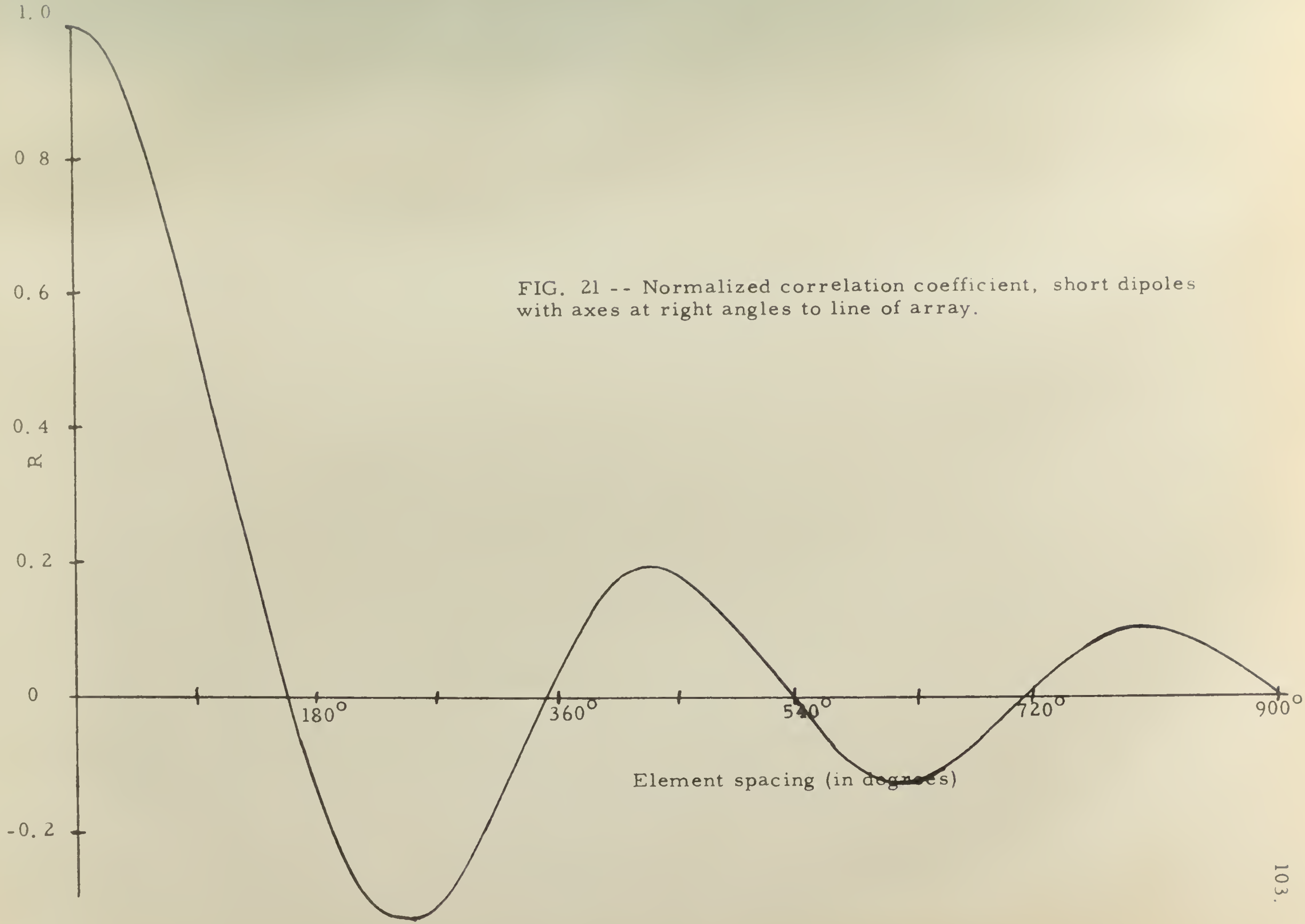


FIG. 21 -- Normalized correlation coefficient, short dipoles with axes at right angles to line of array.

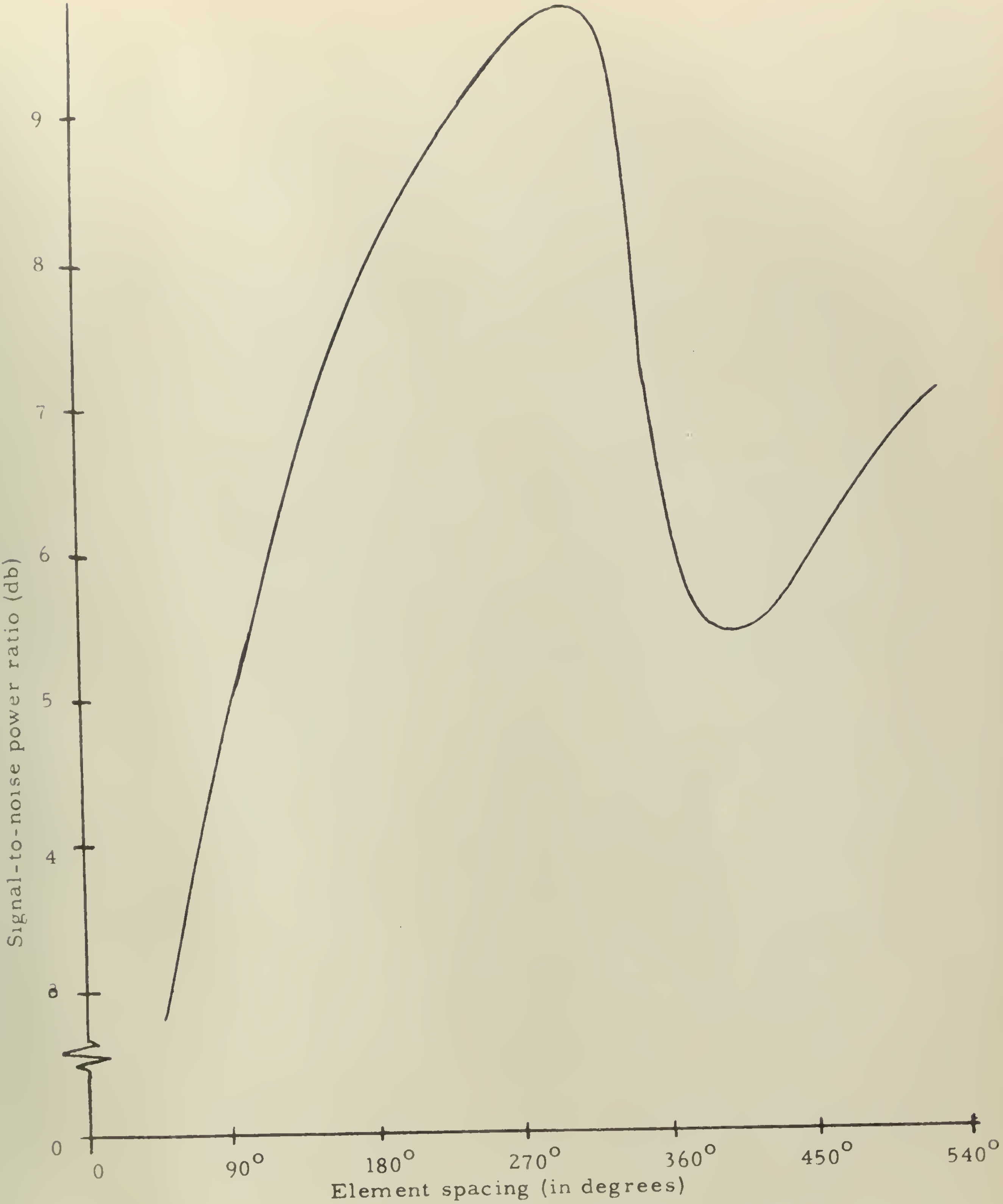


FIG. 22 --Five-element array, uniform spacing, signal-to-noise ratio as a function of element spacing; short dipoles with axes perpendicular to line of array; 0 db corresponds to signal-to-noise ratio on a single antenna.

where, again

$$\Lambda_{3/2}\left(\frac{\omega_0 d}{c}\right) = \Gamma(5/2)\,\frac{J_{3/2}\left(\frac{\omega_0 d}{c}\right)}{\left(\frac{\omega_0 d}{2c}\right)^{3/2}}$$

This normalized correlation coefficient is shown in Fig. 23. The signal-to-noise ratio (in db) of this array of short dipoles is plotted as a function of element spacing in Fig. 24. In this case, maximum of the signal-to-noise ratio is not pronounced and is only slightly greater than that which occurs when the noise is completely uncorrelated. This could be anticipated by observing in Fig. 23 the rapid decrease in magnitude of the correlation coefficient with element spacing.

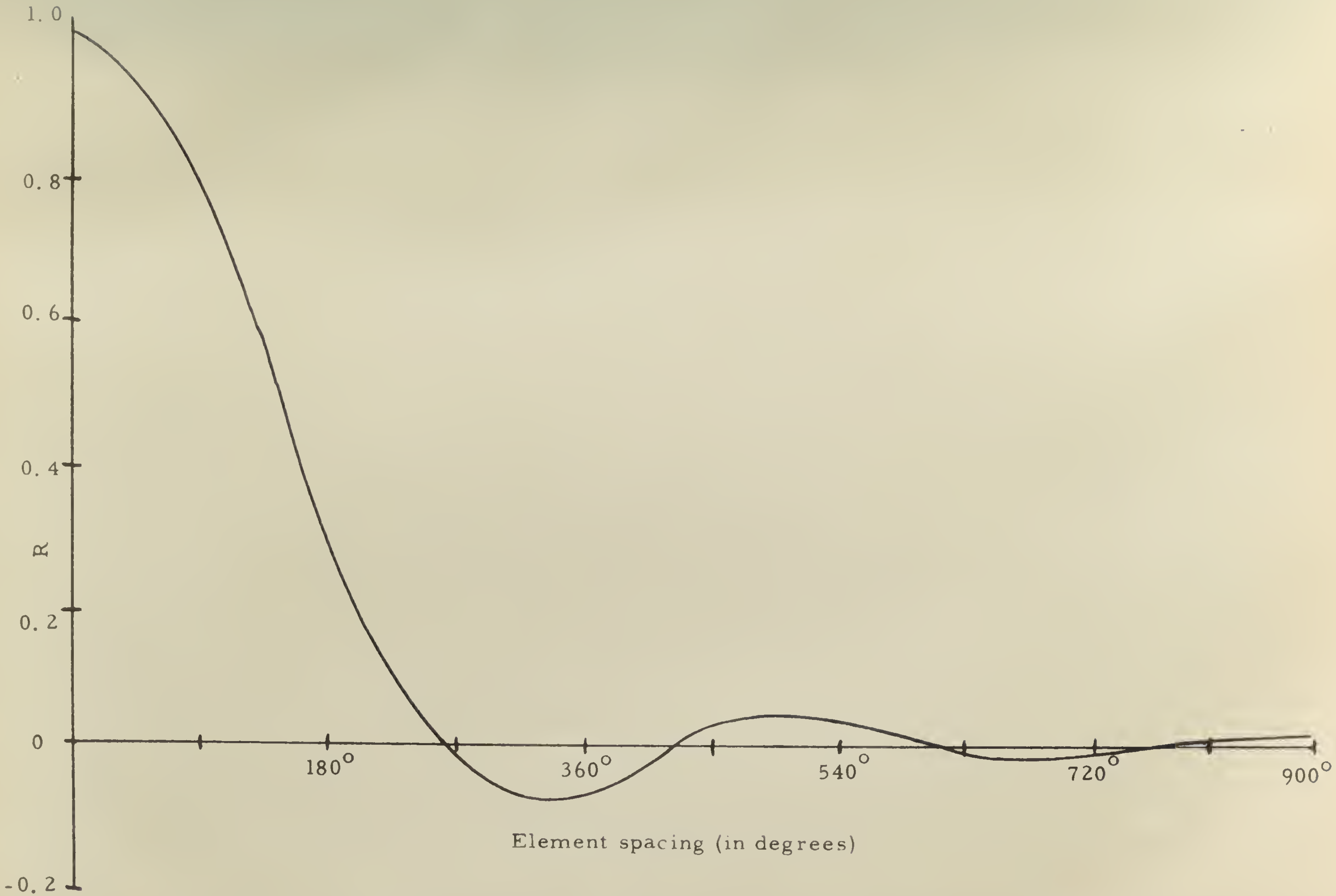


FIG. 23 -- Normalized correlation coefficient; short dipoles with axes in line with line of array.

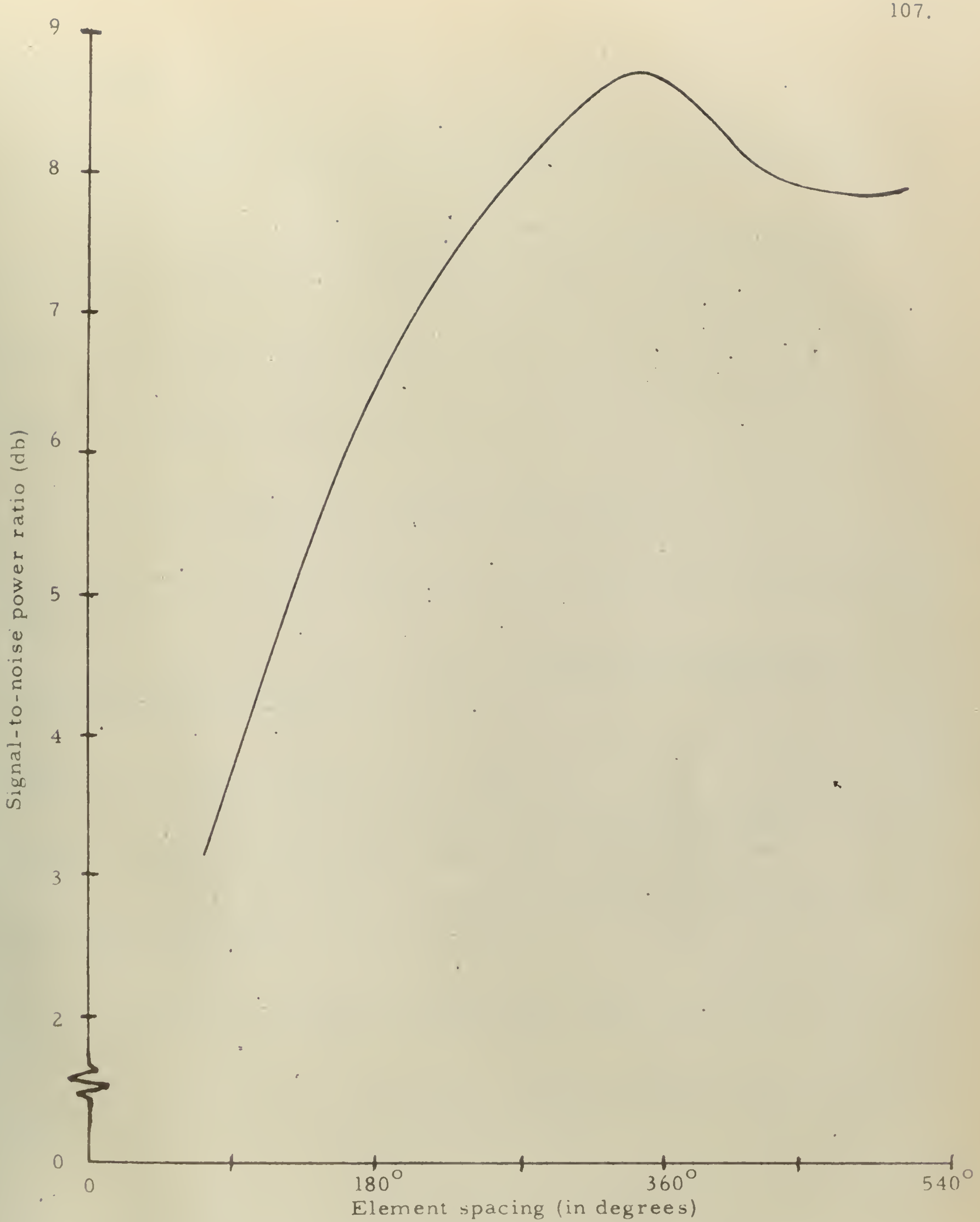


FIG. 24 -- Five-element array, uniform spacing, signal-to-noise ratio as a function of element spacing; short dipoles with axes parallel to line of array; 0 db corresponds to signal-to-noise ratio on a single antenna.

REFERENCES

1. J. J. Faran and R. Hills, "Correlators for Signal Reception," Acoustics Research Laboratory, Harvard University, Tech. Mem. 27, Sept. 15, 1952.

2. J. J. Faran and R. Hills, "The Application of Correlation Techniques to Acoustic Receiving Systems," Acoustics Research Laboratory, Harvard University, Tech. Mem. 28, Nov. 1, 1952.

3. A. Berman and C. S. Clay, "Theory of Time Averaged Product Arrays," J. Acoust. Soc. Amer., Vol. 29, No. 7, p 805, July 1957.

4. J. L. Brown and R. O. Rowlands, "Design of Directional Arrays," J. Acoust. Soc. Amer., Vol. 31, No. 12, p. 1638, Dec., 1959.

5. C. Drane, "Phase Modulated Antennas," Air Force Cambridge Research Center Tech. Rep. 59-138, April 1959.

6. S. O. Rice, "Mathematical Analysis of Random Noise," Bell Syst. Tech. Jour.; Vol. 23, pp 282-332, 1944; Vol. 24, pp 46-156, 1945.

7. W. B. Davenport and W. L. Root, An Introduction to the Theory of Random Signals and Noise, McGraw-Hill Book Co., New York, 1958, pp 158-164.

8. P. E. Green, "The Output Signal-to-Noise Ratio of Correlation Detectors," I.R.E. Trans., Vol. IT-3, No. 1, pp 10-18, March 1957.

9. D. Middleton, An Introduction to Statistical Communication Theory, McGraw-Hill Book Co., New York, 1960, Chap. 16.

10. W. B. Davenport, Jr., "Correlator Errors Due to Finite Observation Intervals," M.I.T. Research Lab. Electronics Tech. Rep. 191, March 1951.

11. R.C. Emerson, "First Probability Densities for Receivers with Square Law Detectors," Jour. Appl. Phys., Vol. 24, No. 9, pp 1168-1176, Sept. 1953.

12. M. A. Meyer and D. Middleton, "On the Distribution of Signals and Noise after Rectification and Filtering," Jour. Appl. Phys., Vol. 25, pp 1037-1052, August 1954.

13. D. G. Lampard, "The Probability Distribution for the Filtered Output of a Multiplier Whose Inputs Are Correlated, Stationary, Gaussian Time-Series," I.R.E. Trans., Vol. IT-2, No. 1, pp 4-11, March 1956.

14. H. Cramer, *Mathematical Methods of Statistics*, Princeton Univ. Press, p 183; 1951.

15. J. I. Marcum, "A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix," The RAND Corp., Research Memorandum 753, July 1, 1948.

thesL644
Correlation processes in antenna arrays.
3 2768 002 11787 1
DUDLEY KNOX LIBRARY